TAKING THE PIRAHÃ SERIOUSLY

FRANCO MONTAGNA, GIULIA SIMI, AND ANDREA SORBI

ABSTRACT. We propose first order formal theories Γ_n, with $n \geq 1$, and $\Gamma = \bigcup_{n\geq 1} \Gamma_n$, which can be roughly described as follows: each one of these theories axiomatizes a bounded universe, with a greatest element, modelled on Sergeyev's so-called grossone; each such theory is consistent if predicative arithmetic $I\Delta_0 + \Omega_1$ is; inside each such theory one can represent (in a weak, precisely specified, sense) the partial computable functions, and thus develop computability theory; each such theory is undecidable; the consistency of Γ_n implies the consistency of $\Gamma_n \cup \{\mathrm{Con}_{\Gamma_n}\}$, where Con_{Γ_n} "asserts" the consistency of Γ_n (this however does not conflict with Gödel's Second Incompleteness Theorem); if $n > 1$, then there is a precise way in which we can say that Γ_n proves that each set has cardinality bigger than every proper subset, although two sets have the same cardinality if and only if they are bijective; if $n > 2$, inside Γ_n there is a precise sense in which we can talk about integers, rational numbers, and real numbers; in particular, we can develop some measure theory; we can show that every series converges, and is invariant under any rearrangement of its terms (at least, those series and those rearrangements we are allowed to talk about); we also give a basic example, showing that even transcendental functions can be approximated up to infinitesimals in our theories: this example seems to provide a general method to replace a significant part of the mathematics of the continuum by discrete mathematics.

1. Introduction

In a series of papers (just to mention some of the papers that are more detailed about foundational aspects, see for instance [1, 2, 3, 4, 5]; se also the book [6]), Sergeyev motivates the introduction of an infinite number, called the *grossone*, and discusses its main properties, as well as some applications, which range from numerical analysis, dynamical systems, even to logic and the theory of computation (see e.g., [7, 8]). The advantages for the use of grossone are clear: cardinalities behave better (for instance, one avoids the well-known Hilbert Grand Hotel paradox), the sum of a series always exists and is invariant under rearrangements of its terms, infinitesimals and infinite elements exist as in nonstandard analysis, etc.

Sergeyev also introduces some basic principles for the theory of the grossone. In listing below these principles, we accompany them by some comments which we hope may serve as a guide to the leading ideas worked out in our paper.

Postulate (1). There exist infinite and infinitesimal objects, but human beings and machines can only execute a finite number of operations.

2010 *Mathematics Subject Classification.* 03C99.

Key words and phrases. Nonstandard models, grossone.

The authors are members of the INDAM-GNSAGA group.

Comment: we accept without comments the first half of this postulate, which is one of the basic principles of nonstandard analysis; we also agree on the second half of the postulate, to the point that in order to deal with infinite and infinitesimal objects, we will propose a very weak and resource bounded formal system.

Postulate (2). We cannot tell what the objects we deal with are, we just shall construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.

Comment: this is a non-technical postulate. One possible interpretation is that the general theory is not a formal axiomatic system, and hence, new tools and new objects may be introduced during the development of the theory. Although we respect this point of view, we do not adopt it in the sequel.

Postulate (3). The principle *the part is less than the whole* is applied to all numbers (finite, infinite and infinitesimal) and to all sets and processes (finite and infinite).

Comment: this is a very interesting point, and we fully support it.

There is yet another, more technical, principle, which is a strengthening of (1):

Postulate (4). There exists an infinite natural number, ①, which is assumed to satisfy all properties which are shared by the usual natural numbers. (This postulate is called by Sergeyev the *Infinite Unit Axiom.*)

Comment: although we accept the existence of ① (called *grossone*), the principle stating that the grossone enjoys all properties which are shared by all natural numbers is, in our opinion, a little bit problematic, as will be explained later.

Although not formulated with reference to a fixed formal language, Postulate (4) is reminiscent of the so called Transfer Principle of nonstandard analysis: any first order property which holds of the usual reals, holds of nonstandard reals as well. However, the theory of grossone has at least one important difference with respect to nonstandard analysis. While in nonstandard analysis the natural numbers constitute an unbounded set, in Sergeyev's theory of the grossone, ① is the greatest natural number and numbers like $\text{①}+1$, $\text{①}+\text{①}$ etc. are no longer natural numbers: the enlarged set of numbers forms the set of the so called *extended natural numbers.*

In our opinion, this principle, according to which natural numbers constitute a bounded set, is an important difference between Sergeyev's theory of the grossone, and nonstandard analysis. Moreover, in [1], the author introduces the interesting example of the Pirahã people, whose members identify all numbers greater than 2, and call them *many.* One way of stretching the Pirahã point of view, would be to suggest that all numbers greater than ① are to be identified. As remarked above, this is not however Sergeyev's point of view: indeed, in the light of Postulate (2), in order to do computations involving ①, Sergeyev introduces the extended natural numbers, which may be much larger than ①. In our opinion, this is quite natural, but in this way the difference between extended natural numbers and nonstandard natural numbers (i.e., the nonstandard natural numbers introduced by nonstandard analysis), becomes very tenuous, for, instead of working in the theory of grossone, we might as well work in the nonstandard natural numbers and stipulate that the grossone is a distinguished infinite natural number.

1.1. **Our proposal.** In this paper, we will follow a different approach. That is, we will follow more closely Sergeyev's initial idea that the grossone is the greatest natural number, and taking the Pirahã point of view seriously, we will identify all numbers greater than the grossone. However, we will require some properties ensuring that the grossone is a really large number.

Our intuitive idea is that the grossone is a number so big that we will never need to use it. The use of the grossone would lead to paradoxes like ① + 1 = ①, which conflicts with the cancellation law (indeed, by cancellation we would get $1 = 0$). But, firstly, counterintuitive results do not mean contradictions, and, secondly, as far as our computations involve numbers smaller than the grossone we do not get counterintuitive results. Finally, as we said before, our axioms will guarantee that the grossone is so big that we have many infinite smaller "substitutes" of it, which may replace it without danger of undesired consequences.

Let us try to be more precise about this point. Our universe will be the numbers smaller than ①, but we require the existence of many infinite natural numbers less than ①, i.e. $\cdots \infty_{-n} \ll \infty_{-2} \ll \infty_{-1} \ll$ ①, where $x \ll y$ means that y is exponentially larger than x. So, we can start from the subuniverse consisting of the numbers less than ∞_{-n}, with a sufficiently large $n > 0$, and then expand the universe if necessary, by using the numbers less than ∞_{-n+k} for some $k < n$. We will see that inside the resulting sets, we can perform part of mathematical analysis, we can represent computable functions, we can code sets of natural numbers and introduce cardinalities so that there is no bijection between a set and a proper subset of it, thus satisfying Postulate (3).

As said before, while our approach fully satisfies Postulates (1) and (3), we do not follow Postulates (2) and (4). As to Postulate (2), we try to do as much mathematics as possible using *only* what in [1] are considered to be the natural numbers. So, our universe constitutes a bounded set with maximum ①.

Moreover, a possible interpretation of Postulate (4) (although this postulate is not formulated with reference to any formal language) is that Sergeyev's extended natural numbers should be viewed to be elementarily equivalent to the finite natural numbers. Indeed, let us formalize Postulate (4) as follows:

> If $\Phi(x)$ is a formula such that for every finite natural number n, $\Phi(n)$ holds in the standard model $\mathbb{N}$ of the natural numbers, then Φ(①) holds in the extended natural numbers.

Then given a sentence φ of arithmetic, the formula $\Phi(x) =: \varphi \wedge x = x$ is satisfied by all natural numbers in the standard model $\mathbb{N}$ if and only if φ is true in $\mathbb{N}$, if and only if it is satisfied by ① in the extended natural numbers (by Postulate (4)), if and only if φ is true in the extended natural numbers. Hence, as far as we can see, the formal theory of the extended natural numbers is not even arithmetical, and its consistency seems to require infinitary methods.

To the contrary, the theory presented in this paper is much weaker: its set of theorems is computably enumerable and its consistency is implied by the consistency of Nelson's Predicative Arithmetic $I\Delta_0+\Omega_1$ (that is, Robinson's Q, plus induction for bounded formulas plus the totality of the function $x^{|y|}$, see [9] or [10]).

Summing up, the basic ideas in this paper are the following:

(a) We try to do mathematics using only the natural numbers in the sense of [1], that is, the numbers smaller than the grossone. We don't use other numbers. The territory beyond the grossone is completely out of reach: "hic sunt leones", as, they say, ancient cartographers used to label in their maps unexplored territories.
(b) We require the presence of infinite natural numbers $\infty_{-1}, \dots, \infty_{-n}$ smaller than the grossone. The numbers smaller than ∞_{-n} play the same role as the numbers smaller than the grossone in [1], and the numbers smaller than the grossone play the role of the extended natural numbers in [1]: see Figure 1.

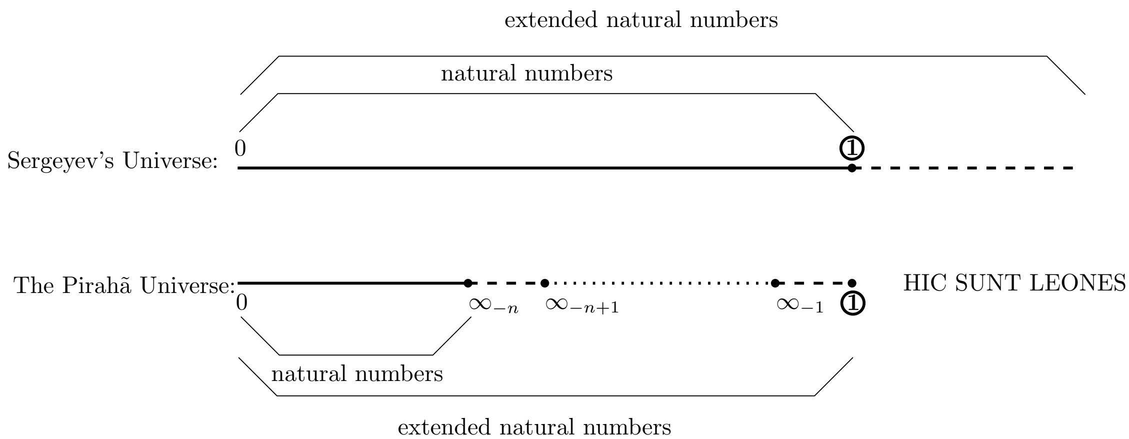

FIGURE 1. Sergeyev vs. Pirahã

(c) Unlike Sergeyev, we give a precise axiomatization of the numbers we are going to use. Indeed, although we believe that formalism does not replace mathematical intuition, we feel safer when we can prove our results in a formal system. Moreover we take care of problems like consistency, and more generally, we try to use the finitistic means of Nelson's Predicative Arithmetic.
(d) Unlike Sergeyev, we do not use things like (finite or infinite) rational numbers or real numbers, we only need natural numbers. The natural numbers smaller than ∞_{-n} for n sufficiently large are our starting point. These numbers are iterated logarithms of other numbers and hence their iterated exponentials exist. Integers are coded by natural numbers less than $2 \cdot \infty_{-n}^2$, and hence, less than ∞_{-n+1}. We can approximate real numbers by rational numbers with denominator ∞_{-n}. Since their denominator is fixed, we can represent these numbers using their numerator only, which will be an integer in $[-\infty_{-n}^2, \infty_{-n}^2]$, and hence may be coded by a natural number less than ∞_{-n+1}. In particular, the numbers we need are all less than the grossone. But if we want to code subsets of $[0, \infty_{-n+1}]$ or functions on $[0, \infty_{-n+1}]$ we need larger natural

numbers, that is, natural numbers in $[0, \infty_{-n+2}]$. In any case, we can use the natural numbers less than ∞_{-n} with n sufficiently large, as ground natural numbers and then use larger natural numbers to represent infinite or infinitesimal reals, or sets, or functions on them.

(e) The weakness of our theory allows us to isolate those operations which are feasible, that is, polynomial time. In developing our setting our first reference has been the so-called *bounded arithmetic*, a formal system which was introduced by Nelson [9] for foundational purposes, aiming at a predicative arithmetic; interest in bounded arithmetic subsequently grew up, due to the work of Buss [11], who gave a proof theoretic characterization of the polynomial time computable functions. (Wilkie and Paris [10] have also shown that Gödel's incompleteness results hold in bounded arithmetic.) Nevertheless, contrary to what happens in bounded arithmetic, where some functions like the exponential function are partial, in our case, exponential and factorial are total, but they may take value ①. The exponential and factorial functions are fundamental in our setting, in that they allow us to jump to the next infinite number, which is not reachable by polynomial time operations only.

The paper is organized as follows. In Section 2 we exhibit the axioms of our theory. We will present many different theories Γ_n, all interpretable in $I\Delta_0 + \Omega_1$, as well as their union Γ. In Section 3 we prove that the computable functions are definable, in a weak sense, in Γ_n. In Section 4 we prove that Γ_n is interpretable in $I\Delta_0 + \Omega_1$, and in Section 5 we prove that some elementary mathematics can be performed inside Γ_n. In particular, inside Γ_n we can construct structures which are similar to the integers and to the rationals, we can code bounded sets, and we can do some elementary calculus. We also give a basic example, showing that even transcendental functions can be approximated up to infinitesimals in our system.

As far as we know, the setting we propose, as well as the basic technical results, are new, and have not appeared earlier in the literature.

2. Axiomatization

We will present a sort of predicative theory of the grossone. More precisely, for all n we have a theory Γ_n, where the choice of n depends on how much mathematics we want to do in our theory. The whole theory Γ will be the union of all Γ_n.

Before giving the list of axioms, we will present the leading ideas behind our theories. The universe we are going to describe is obtained as follows. Take a nonstandard model $\mathbb{M}$ of $I\Delta_0 + \Omega_1$: since all polynomial time functions are definable and provably total in $I\Delta_0 + \Omega_1$, we can expand $\mathbb{M}$ to a model equipped with all polynomial time functions in the language of Γ_n, which is an expansion of the language of $I\Delta_0 + \Omega_1$. In particular, $\mathbb{M}$ is closed under the functions $|y|!$ and $x^{|y|}$, where $|z|$ denotes the length of the binary representation of z minus 1. In order to obtain a model of Γ_n, let N be a nonstandard element of the model: define by induction $N_0 = N$, $N_{i+1} = ||N_i||$ (here $||\_||$ is the function $|(|\_|)|$, i.e., $|\_|$ is applied twice), and let $\infty_{-n} = N_n$; then, for $0 < k < n$, let $\infty_{-n+k} = (\infty_{-n+k-1}!)^{\infty_{-n+k-1}}$, and $① = (\infty_{-1}!)^{\infty_{-1}}$.

Note that all these elements exist in the model, by the closure properties of $\mathbb{M}$, and the fact that $\infty_{-n+k} \leq N_{n-k} = ||N_{n-k-1}||$ (as for almost all x, $(||x||!)^{||x||} < x$), which is easily seen to imply that $\infty_{-n+k} = |z|$, for some z. The universe of the model $\mathbb{K}$ we are going to define, is the interval $[0, ①]$. For any polynomial time operation $f(x_1, \ldots, x_n)$ of $\mathbb{M}$, and for all $a_1, \ldots, a_n \leq ①$, set

$$f^{\mathbb{K}}(a_1, \ldots, a_n) = \min\{f(a_1, \ldots, a_n), ①\}.$$

In our model $\mathbb{K}$, we want also to somehow represent the truncations of some functions that are not polynomial time, like the exponential x^y and the factorial $y!$. The idea is the following. The functions $\mathrm{Exp}_{|z|}(x, y) = x^{\min\{y,|z|\}}$ and $y!_{|z|} = (\min\{y, |z|\})!$ are both polynomial time, and hence, they are total in $\mathbb{M}$, and we can thus take their truncations to ①, to represent them in $\mathbb{K}$. In particular, $\mathrm{Exp}_{|①|}(x, y)$ and $y!_{|①|}$ are total in $\mathbb{M}$, and we take

$$\mathrm{Exp}^{\mathbb{K}}(x, y) = \min\{\mathrm{Exp}_{|①|}(x, y), ①\} \qquad y!^{\mathbb{K}} = \min\{y!_{|①|}, ①\}.$$

This ends the intuitive description of our theories. Now we come to a formal presentation.

2.1. **Language and axioms.** Both the language of Γ_n and the language of Γ contain symbols for some polynomial time computable functions, including the successor function S, the functions $+$, $\cdot$, $|x|$, (recall that $|x|$ is interpreted by the length of the binary representation of the number x minus 1), the functions $\mathrm{Exp}_{|z|}(x, y)$, and $y!_{|z|}$, the predecessor function P, the function $x/2 = \max\{y : 2y \leq x\}$, the truncated difference $\ominus$, where $x \ominus y = \max\{x - y, 0\}$, the pairing function $\langle x, y\rangle$ and its projections π_1, π_2, see e.g. [12] . The language has also the following closure properties: for every $n + 1$-ary term f, our language contains the symbol Σ_f, representing the function

$$g(x_1, \ldots, x_n, i, z) = \sum_{j=0}^{\min\{|z|, i\}} f(x_1, \ldots, x_n, j);$$

and for every $n + 1$-ary term $f(x_1, \ldots, x_n, y)$, the language contains a function symbol $\mu z \leq |y|.f(x_1, \ldots, x_n, z) \equiv 0$, representing the function

$$h(x_1, \ldots, x_n, y) = \begin{cases} \min\{z \leq |y| : f(x_1, \ldots, x_n, z) = 0\}, & \text{if such a } z \text{ exists} \\ |y| + 1 : & \text{otherwise.} \end{cases}$$

Note that both $\Sigma_f(x_1, \ldots, x_n, i, z)$ and $\mu z \leq |y|.f(x_1, \ldots, x_n) \equiv 0$ represent polynomial time functions when f is polynomial time.

Finally, the language of Γ contains the symbols $=$ (equality) and $<$, ($x \leq y$ stands for "$x = y \vee x < y$"), as well as the constants

$$0, ①, \infty_{-1}, \infty_{-2}, \ldots, \infty_{-n}, \ldots.$$

Of the constants $\infty_{-1}, \infty_{-2}, \ldots, \infty_{-n}, \ldots$, the language of Γ_n has only the symbols ∞_{-i} for $i = 1, \ldots, n$.

Definition 2.1. The formulas (terms, respectively) of the language, containing neither ① nor any ∞_{-i}, with obvious suggestion will be referred to as the *arithmetical formulas* (*arithmetical terms*, respectively).

Notice that arithmetical formulas and terms can be interpreted in the standard model $\mathbb{N}$ of the natural numbers.

As usual we define the *numerals*, i.e. the closed terms that are taken to represent the finite natural numbers in our language: for every finite natural number m, let $\overline{m}$ be the closed term defined as $S(\dots(S(0)\dots)$, m times. However, we go for the lazy option in this regard, and we warn the reader to read carefully the following remark on notation.

Warning. With the intent of simplifying the typographical setting of mathematical formulas, in the rest of the paper we will not typographically distinguish between a finite natural number m, and its corresponding numeral $\overline{m}$, thus always writing m and leaving to the reader to spot which one is which.

The axioms. We are now going to list the axioms of Γ and of Γ_n. Most of them are just obvious properties of elementary arithmetic: the reader may perhaps just skip them, and concentrate on those axioms that more directly involve ①, or the constants ∞_{-1}, $i \geq 1$. We make no claim of course about the axioms being independent: our long, and undoubtedly tedious, list of axioms has only the purpose of making easily and immediately available all the principles that are needed to develop our system.

(1) Axioms for the successor S and for the predecessor P (we write $S(x) = x + 1$).
- (1a) $\neg(x + 1 = 0)$.
- (1b) $x + 1 = y + 1 \to (x = ① \vee y = ① \vee x = y)$.
- (1c) $① + 1 = ①$.
- (1d) $x = 0 \vee \exists y(y + 1 = x)$.
- (1e) $P(①) < ①$.
- (1f) $P(0) = 0$.
- (1g) $x < ① \to P(x + 1) = x$.
- (1h) $0 < x \to P(x) + 1 = x$.

(2) Axioms for the functions of our language. In particular:
- (2a) $x + 0 = 0 \qquad x + (y + 1) = (x + y) + 1$.
- (2b) $x \cdot 0 = 0 \qquad x \cdot (y + 1) = x \cdot y + x$.
- (2c) $|0| = |1| = 0$.
- (2d) $(x + P(x) < ① \wedge 0 < x) \to (|2 \cdot x| = |x| + 1 \wedge |2 \cdot x + 1| = |x| + 1)$.
- (2e) $z \leq |x| \to \exists u(|u| = z)$.
- (2f) $x/2 + P(x/2) < x \qquad x/2 + x/2 \leq x \qquad x \leq x/2 + x/2 + 1$.
- (2g) $(x + y) \cdot (x + y + 1) + 2 \cdot x = 2 \cdot \langle x, y\rangle$.
- (2h) $\langle x, y\rangle < ① \to (\pi_1(\langle x, y\rangle) = x \wedge \pi_2(\langle x, y\rangle) = y) \qquad x < ① \to \langle \pi_1(x), \pi_2(x)\rangle = x$.
- (2k) $\sum_f(x_1, \dots, x_n, 0, z) = f(x_1, \dots, x_n, 0)$.
- (2i) $y < |z| \to \sum_f(x_1, \dots, x_n, y + 1, z) = \sum_f(x_1, \dots, x_n, y, z) + f(x_1, \dots, x_n, y + 1)$.
- (2j) $(|z| \leq y) \to \sum_f(x_1, \dots, x_n, y + 1, z) = \sum_f(x_1, \dots, x_n, y, z)$.
- (2l) $\forall w(\mu z \leq |y|.f(x_1, \dots, x_n, z) \equiv 0 = w \leftrightarrow$
  $(\forall w' < w)\neg f(x_1, \dots, x_n, w') = 0 \wedge (w = |y| + 1 \vee (w \leq |y| \wedge f(x_1, \dots, x_n, w) = 0))$.
- (2m) $x \ominus 0 = x \qquad x \ominus (y + 1) = P(x \ominus y)$.
- (2n) $\mathrm{Exp}_{|z|}(x, 0) = 1 \qquad y < |z| \to \mathrm{Exp}_{|z|}(x, y + 1) = \mathrm{Exp}_{|z|}(x, y) \cdot x$.

(2o) $|z| \leq y \rightarrow \mathrm{Exp}_{|z|}(x, y+1) = \mathrm{Exp}_{|z|}(x, y)$.
(2p) $0!_{|z|} = 1 \qquad y < |z| \rightarrow (y+1)!_{|z|} = y!_{|z|} \cdot (y+1)$.
(2q) $|z| \leq y \rightarrow (y+1)!_{|z|} = y!_{|z|}$.

(3) Order axioms for $\leq$, namely:
(3a) $x \leq x$.
(3b) $(x \leq y \wedge y \leq x) \rightarrow x = y$.
(3c) $(x \leq y \wedge y \leq z) \rightarrow x \leq z$.
(3d) $x \leq y \vee y \leq x$.
(3e) $0 \leq x \wedge x \leq ①$.
(3f) $x \leq y \rightarrow x + z \leq y + z$.
(3g) $x \leq y \rightarrow x \cdot z \leq y \cdot z$.
(3h) $x \leq y \rightarrow \exists z(x + z = y)$.
(3i) $x \leq x + 1 \wedge (x < ① \rightarrow x < x + 1)$.
(3l) $x \leq y \rightarrow (x = y \vee x + 1 \leq y)$.

(4) Full induction, that is, induction for *all* first-order formulas.

(5) Axioms for the new constants. Let us write x^y and $y!$ for $\mathrm{Exp}_{|①|}(x, y)$ and for $y!_{|①|}$, respectively. Then we introduce the following axioms:
(5a) $(\infty_{-1}!)^{\infty_{-1}} = ①$.
Finally, for $k = 1, 2, \ldots$, Γ has the axioms:
(5bk) $(\infty_{-k-1}!)^{\infty_{-k-1}} = \infty_{-k}$.
(5ck) $\infty_{-k} \leq |①|$.

(6) Axioms on finite natural numbers:
(6a) Γ has axioms $m < \infty_{-i}$, for all finite natural numbers $m = 0, 1, \ldots$, and for all $i \geq 1$.

(7) Axioms distinguishing Γ_n from Γ:
(7a) Γ_n has all axioms $m < \infty_{-i}$, for all finite natural numbers $m = 0, 1, \ldots$, and for all $1 \leq i \leq n$.
(7b) Γ_n has axioms (5bk) and (5ck) only for $k = 1, \ldots, n-1$.

Note that Γ_n is a subtheory of Γ.

Lemma 2.2. *The following can be proved in Γ_n, if $1 \leq k \leq n$ (where we assume $\infty_0 = ①$):*

(1) $\infty_{-k} < |\infty_{-k+1}| < \infty_{-k+1}$*;*
(2) $\forall x, y\,(0 < x \wedge x, y \leq \infty_{-k} \rightarrow x^y | \infty_{-k+1})$*, where* $u|v$ *("*u divides v*") stands for* $\exists z\,(v = u \cdot z)$*;*
(3) $\infty_{-k}! < \infty_{-k}^{\infty_{-k}} < \infty_{-k+1}$.

Proof. It is an immediate consequence of the axioms that Γ_n, when restricted to elements below $|①|$, proves the familiar arithmetical properties, such as commutativity, associativity, distributivity, etc., even when dealing with numbers of the form ∞_{-k}: the main tool for checking these properties is of course induction.

To prove (1), first notice that, by induction on y, it is easy to see that Γ_n proves

$$(0 < y < |①|) \rightarrow y = |2^y|,$$

where $2^y =: \mathrm{Exp}_{|①|}(2,y)$. Now, $\infty_{-k} = |2^{\infty_{-k}}|$; on the other hand, we have that $\infty_{-k}! = 2 \cdot q$, for some $q > 2$; hence by simple, easily verified properties:

$$2^{\infty_{-k}+1} = 2^{\infty_{-k}} \cdot 2 < 2^{\infty_{-k}} \cdot q^{\infty_{-k}} = (2 \cdot q)^{\infty_{-k}} = \infty_{-k}!^{\infty_{-k}} = \infty_{-k+1};$$

hence $\infty_{-k} < |\infty_{-k+1}|$. Moreover by induction on x, one sees that $0 < x \to |x| < x$, thus also $|\infty_{-k+1}| < \infty_{-k+1}$. This proves (1). Similarly, $\infty_{-1} < |①| < ①$.

To show (2), from $0 < x$ and $x, y \leq \infty_{-k}$ one first infers that $\infty_{-k}! = x \cdot q$, for some q, hence $\infty_{-k}!^y = x^y \cdot q^y$, i.e. $x^y | \infty_{-k}!^y$, but $\infty_{-k}!^y | \infty_{-k}!^{\infty_{-k}}$, hence $x^y | \infty_{-k}!^{\infty_{-k}} = \infty_{-k+1}$.

Finally, to show (3), notice that $\infty_{-k}! < \infty_{-k}^{\infty_{-k}}$; on the other hand, as $\infty_{-k} < \infty_{-k}!$, it follows that

$$\infty_{-k}^{\infty_{-k}} < \infty_{-k}!^{\infty_{-k}} = \infty_{-k+1}.$$

□

It is easy to see that full induction (4) is equivalent to the *Least Element Principle*:

(4bis) (Least Element Principle) for every formula φ (with free variables $x_1, \ldots, x_m, x$),

$$(\exists x)(\varphi(x) \to (\exists y)(\varphi(y) \wedge (\forall z < y)\neg\varphi(z))$$

Notation. Given a structure $\mathbb{M}$ of our language, and a function symbol f, we denote by $f^{\mathbb{M}}$ the function that interprets f in $\mathbb{M}$. By induction on the complexity of a term t, this defines also the function $t^{\mathbb{M}}$. If $a_1, \ldots, a_m$ are elements of $\mathbb{M}$, and $\mathbb{M}$ satisfies a formula $\psi(v_1, \ldots, v_m)$, when we interpret the variables $v_1, \ldots, v_m$ with $a_1, \ldots, a_m$, respectively, we write $\mathbb{M} \models \psi(a_1, \ldots, a_m)$.

Definition 2.3. Given a model $\mathbb{M}$ of Γ, or Γ_n, a function symbol $f(x_1, \ldots, x_m)$, and an element $a \in \mathbb{M}$, we write $f^{\mathbb{M}}_{\leq a}$ to denote the function $f^{\mathbb{M}}_{\leq a} : ([0,a])^m \longrightarrow [0,a]$,

$$f^{\mathbb{M}}_{\leq a}(a_1, \ldots, a_m) = \min\{f^{\mathbb{M}}(a_1, \ldots, a_m), a\}.$$

If t is a term, $t =: f(t_1, \ldots, t_m)$, and we have interpreted each constant symbol c with an element $c^{\mathbb{M}}_{\leq a} \leq a$, then by induction on the complexity of t, one defines in the obvious way the function $t^{\mathbb{M}}_{\leq a} =: f^{\mathbb{M}}_{\leq a}((t_1)^{\mathbb{M}}_{\leq a}, \ldots, (t_m)^{\mathbb{M}}_{\leq a})$. We say in this case that $t^{\mathbb{M}}_{\leq a}$ is *defined from* $t^{\mathbb{M}}$ *by truncation to* a.

We observe that in fact every truncation of a term is in turn a term.

Lemma 2.4. *For each term* $t(x_1, \ldots, x_m)$ *there exists a term* $t_{tr}(x_1, \ldots, x_m, z)$ *such that, for all* $a_1, \ldots, a_m \leq a$,

$$t^{\mathbb{M}}_{\leq a}(a_1, \ldots, a_m) = t^{\mathbb{M}}_{tr}(a_1, \ldots, a_m, a).$$

Proof. It follows by induction on the complexity of the term t, by first observing that if t is a function symbol, say $t =: f(x_1, \ldots, x_m)$ then (provably in Γ, and in Γ_n)

$$\min\{f(x_1, \ldots, x_m), z\} = f(x_1, \ldots, x_m) \ominus (f(x_1, \ldots, x_m) \ominus z),$$

thus we can take $f_{tr}(x_1, \ldots, x_m, z) =: f(x_1, \ldots, x_m) \ominus (f(x_1, \ldots, x_m) \ominus z)$: we have

$$f^{\mathbb{M}}_{\leq a}(a_1, \ldots, a_m) = f^{\mathbb{M}}_{tr}(a_1, \ldots, a_m, a),$$

for all $a_1, \dots, a_m \leq a$. □

The next theorem shows how one can define models of Γ (Γ_n), taking as universes suitable initial segments of already given models of Γ (Γ_n).

Theorem 2.5. *The following hold:*

(1) Let $\mathbb{M}$ be a model of Γ. For any $n \geq 1$, let $\mathbb{K}$ be the structure defined as follows:
   (a) the universe is $[0, \infty_{-n}]$;
   (b) $f^{\mathbb{K}} =: f^{\mathbb{M}}_{\leq \infty_{-n}}$ for every function symbol f;
   (c) ① interpreted as ∞_{-n}, and ∞_{-k} interpreted as ∞_{-n-k};
   (d) $=$ and $<$ (for which we do not use any typographical distinction) are the obvious restrictions to $\mathbb{K}$.

   Then $\mathbb{K}$ is a model of Γ.

(2) Let $\mathbb{M}$ be a model of Γ_{n+k} with $n, k > 0$. Let $\mathbb{M}_k$ be the structure with the interval $[0, \infty_{-k}]$ as universe, where grossone ① is interpreted as ∞_{-k}, the constant symbol ∞_{-i} is interpreted as ∞_{-k-i}, for $i = 1, \dots, n$, and the function symbols are interpreted by truncation to ∞_{-k}, following the pattern of the previous item. Then $\mathbb{M}_k$ is a model of Γ_n.

Proof. Let us examine (1): item (2) is proved similarly. Let $\mathbb{K}$ be a structure, with universe $[0, \infty_{-n}]$, obtained by a model $\mathbb{M}$ of Γ, as in the statement of the theorem: we want to show that $\mathbb{K}$ is a model of Γ as well. As to induction, i.e. axiom scheme (4), assume that $\psi(x_1, \dots, x_m, u)$ is a formula, $a_1, \dots, a_m \in \mathbb{K} = [0, \infty_{-n}]$, and there is a $b \in \mathbb{K}$ such that $\psi(a_1, \dots, a_m, b)$ holds in $\mathbb{K}$: we want to show in this case that there is a minimum b with this property. To this purpose, we transform ψ into a formula ψ', performing the following steps:

- we first transform ψ into a logically equivalent formula having all subformulas of the form $x \leq y$, or $y = f(x_1, \dots, x_m, u)$ for some function symbol f and variables among $x_1, \dots, x_m, u$;
- we replace each function symbol $f(x_1, \dots, x_m, u)$ by $f_{tr}(x_1, \dots, x_m, u, z)$ (as defined in Lemma 2.4), where z is some new variable;
- we replace every subformula of the form $\exists x\, \gamma$, by $\exists x\, (x \leq z \wedge \gamma)$ and every subformula of the form $\forall x\, \gamma$, by $\forall x\, (x \leq z \rightarrow \gamma)$.

An easy computation shows that

$$\mathbb{K} \models \psi(a_1, \dots, a_m, b) \Leftrightarrow \mathbb{M} \models \psi'(a_1, \dots, a_m, b, \infty_{-n}).$$

Since $\mathbb{M}$ is a model of induction, there is a minimum $b_0 \leq \infty_{-n}$ in $\mathbb{M}$ such that $\mathbb{M} \models \psi'(a_1, \dots, a_m, b_0, \infty_{-n})$. But then, b_0 is the minimum b such that $\mathbb{K} \models \psi(a_1, \dots, a_m, b)$.

As to the remaining axioms, it is trivial to show that most of them are valid in $\mathbb{K}$. Some of the axioms in (2), in particular those involving recursion, deserve perhaps some attention, due to the fact that in $\mathbb{K}$ functions are interpreted by truncation. By way of exemplification, let us consider for instance axioms (2i) and (2n).

(Axiom (2i).) If $a_1, \dots, a_m, b, c \le \infty_{-n}$, and $b < |c|$, we first observe that (notice that $b+1$ gives the same element whether we work in $\mathbb{M}$ or in $\mathbb{K}$),

$$(\Sigma_f)^{\mathbb{K}}(a_1, \dots, a_m, b+1, c) = (\Sigma_f)^{\mathbb{M}}_{\le \infty_{-n}}(a_1, \dots, a_m, b+1, c),$$

and, by validity of the axiom in $\mathbb{M}$,

$$(\Sigma_f)^{\mathbb{M}}(a_1, \dots, a_m, b+1, c) = (\Sigma_f)^{\mathbb{M}}(a_1, \dots, a_m, b, c) +^{\mathbb{M}} f^{\mathbb{M}}(a_1, \dots, a_m, b+1).$$

The claim now follows by a case by case inspection, distinguishing whether $(\Sigma_f)^{\mathbb{M}}(a_1, \dots, a_m, b+1, c) < \infty_{-n}$, or $(\Sigma_f)^{\mathbb{M}}(a_1, \dots, a_m, b+1, c) \ge \infty_{-n}$: in the former case notice that $+^{\mathbb{M}} = +^{\mathbb{K}}$, as we are $< \infty_{-n}$; in the latter case it might be convenient to further distinguish whether or not $(\Sigma_f)^{\mathbb{M}}(a_1, \dots, a_m, b, c) < \infty_{-n}$ or $f^{\mathbb{M}}(a_1, \dots, a_m, b+1) < \infty_{-n}$.

(Axiom (2n).) Let us now show that $\mathbb{K}$ satisfies (2n). To this end, let $a, b, c \in \mathbb{K}$, with $b < |c|$; assume also $a > 0$: the case $a = 0$ can be easily treated separately. To simplify expressions, let us write $f(x, y, z) =: \mathrm{Exp}_{|z|}(x, y)$: we want to show that

$$f^{\mathbb{K}}(a, b+1, c) = f^{\mathbb{K}}(a, b, c) \cdot^{\mathbb{K}} a.$$

(Once again, $b+1$ gives the same element whether we work in $\mathbb{K}$ or in $\mathbb{M}$.) The claim follows again by a case by case inspection, distinguishing whether $f^{\mathbb{M}}(a, b, c) \cdot^{\mathbb{M}} a \ge \infty_{-n}$ (in this case, further distinguishing whether or not $f^{\mathbb{M}}(a, b, c) \ge \infty_{-n}$) or $f^{\mathbb{M}}(a, b, c) \cdot^{\mathbb{M}} a < \infty_{-n}$. □

2.2. **A model of Γ_n.** We present the construction of a model of Γ_n, based on the notion of ultraproduct (for ultraproducts, and their use in model theory, see e.g. [13]). For $k = 2, 3 \dots$, define $k_0 = k$, $k_{i+1} = (k_i!)^{k_i}$, and for every $n \ge 1$, let $\mathbb{M}_n^k$ be the structure whose domain is $[0, k_n]$, where the constant ① is interpreted as k_n, and ∞_{-i} is interpreted as k_{n-i}, equipped with all operations of Γ_n truncated to k_n. Clearly, $\mathbb{M}_n^k$ is not a model of Γ_n, because any model of Γ_n is infinite. However:

Theorem 2.6. *For any fixed $n \ge 1$, any ultraproduct $\mathbb{M}_n^*$ of the $\mathbb{M}_n^k$, $k = 2, \dots,$ modulo a non-principal ultrafilter U on $\mathbb{N}$, is a model of Γ_n.*

Proof. Given a sequence $d \in \prod_{2 \le k} \mathbb{M}_n^k$, let d_U denote the equivalence class of d in the ultraproduct determined by the ultrafilter U. Thus, for instance, ① in $\mathbb{M}_n^*$ is interpreted by $\langle 2_n, 3_n, 4_n, \dots \rangle_U$, and ∞_{-i} by $\langle 2_{n-i}, 3_{n-i}, 4_{n-i}, \dots \rangle_U$, for $0 < i \le n$. We first show that the ultraproduct satisfies induction: to this end, assume that $\langle b_2, b_3, b_4, \dots \rangle_U \in \mathbb{M}_n^*$ satisfies in $\mathbb{M}_n^*$ a formula $\psi(x)$. Then the set $K = \{k : \mathbb{M}_n^k \models \psi(b_k)\}$ lies in U. For every $k \in K$, let c_k be the least element such that $\mathbb{M}_n^k \models \psi(c_k)$. Let d be any sequence such that $d_k = c_k$ for $k \in K$. Then it is easy to see that d_U is the least element in $\mathbb{M}_n^*$ which satisfies $\psi(x)$ in $\mathbb{M}_n^*$. One easily checks that (7a) holds: indeed, for all $m \in \mathbb{N}$ and $i \le n$, there exists $k' \in \mathbb{N}$ such that, for every $k \in \mathbb{N}$, $k \ge k'$, we have that $m < k_{n-i}$, thus $\mathbb{M}_n^* \models m < \infty_{-i}$, as m is interpreted in $\mathbb{M}_n^*$ by $\langle d \rangle_U$, where d is any sequence which takes eventually value m. Easy calculations show that also the remaining axioms are satisfied in $\mathbb{M}_n^*$. □

Any model of the form $\mathbb{M}_n^*$ will be called a *natural model of* Γ_n. Not all models of Γ_n are elementarily equivalent to a natural model. Indeed, consider a universal arithmetical sentence $\varphi =: \forall x_1 \dots \forall x_n \varphi_0(x_1, \dots, x_n)$, with φ_0 quantifier free. Let $t_1, \dots, t_k$ be the terms occurring in φ. Finally, let φ° be the sentence

$$\forall x_1 \dots \forall x_n (t_1 = ① \vee \dots \vee t_k = ① \vee \varphi_0).$$

It is not hard to prove that if $\mathbb{N} \models \varphi$ then for all k, $\mathbb{M}_n^k \models \varphi^\circ$, and that if $\mathbb{N} \not\models \varphi$ then for k sufficiently large, $\mathbb{M}_n^k \not\models \varphi^\circ$. Hence, $\mathbb{N} \models \varphi$ if and only if $\mathbb{M}_n^* \models \varphi^\circ$. Since the set of all universal sentences which are true in the standard model of the natural numbers is not computably enumerable, we obtain:

Theorem 2.7. *The set of all sentences which are valid in all natural models of* Γ_n *is not computably axiomatizable.*

Proof. The above argument shows that the set of all universal sentences which are true in the standard model of the natural numbers is many-one reducible (via the assignment $\varphi \mapsto \varphi^\circ$) to the set of sentences that are true in all natural models. Since the former set is not computably enumerable, so is not the latter one. In particular the set of all universal sentences which are true in all natural models of Γ_n, is not computably axiomatizable. □

By the Completeness Theorem, it follows that not all models of Γ_n are elementarily equivalent to a natural model.

3. Recursion Theory in Γ_n

In this section we prove that the computable functions are representable in Γ_n, in a weak sense to be specified later. Throughout the section we assume consistency of Γ_n and Γ, although the issue of consistency will be addressed in Section 4.

Lemma 3.1. *The standard model* $\mathbb{N}$ *of the natural numbers is an initial segment of any model of* Γ_n.

Proof. Let $\mathbb{M}$ be a model of Γ_n. Let $\iota : \mathbb{N} \longrightarrow \mathbb{M}$, $\iota(m) = m^{\mathbb{M}}$. By induction on m, it is not difficult to see that, for all $k, m \in \mathbb{N}$,

$$k < m \Leftrightarrow \mathbb{M} \models k < m.$$

On the other hand, for each $a \in \mathbb{M}$, and $k \in \mathbb{N}$, $\mathbb{M} \models a \leq k$ implies that there exists $m \in \mathbb{N}$ such that, in $\mathbb{M}$, $a = m^{\mathbb{M}}$. This easily follows from that fact that, for every $m \in \mathbb{N}$,

$$\Gamma_n \vdash x \leq m \leftrightarrow x = 0 \vee x = 1 \vee \dots \vee x = m, \tag{1}$$

as one can check by induction on m. It follows that ι embeds $\mathbb{N}$ into $\mathbb{M}$ as an initial segment, as desired. □

Definition 3.2. Let $\mathbb{M}$ be a model of Γ_n. An element $a \in \mathbb{M}$ is said to be *finite* if a is in the range of the embedding ι, defined in the previous lemma; otherwise a is *infinite*.

Definition 3.3. The class Δ_0 of formulas is defined by induction as follows:

(1) each atomic formula is in Δ_0;
(2) Δ_0 is closed under propositional connectives;

(3) if $\varphi \in \Delta_0$, v is a variable, and t is a term in which v does not occur, then $(\exists v < t)\,\varphi$ and $(\forall v < t)\,\varphi$ are in Δ_0. (We recall that $(\exists v < t)\varphi$ is a shorthand for $\exists v(v < t \wedge \varphi)$; likewise $(\forall v < t)\,\varphi$ stands for $\forall v\,(v < t \rightarrow \varphi)$; similar meanings are given to the abbreviations $(\exists v \leq t)\varphi$ and $(\forall v \leq t)\,\varphi$: of course Δ_0 is also closed under bounded quantifiers of the form $(\exists v \leq t)\varphi$ and $(\forall v \leq t)\,\varphi$.)

The class Σ_1 of formulas consists of all formulas of the form $\exists v_1 \ldots \exists v_m \varphi$ where $\varphi \in \Delta_0$.

Lemma 3.4. *The following hold, where we limit ourselves to sentences in Δ_0, or in Σ_1 which are just arithmetical sentences:*

(a) If φ is a Δ_0 sentence then φ is true in the standard model $\mathbb{N}$ of the natural numbers if and only if it is provable in Γ_n.
(b) Let φ be a Σ_1 sentence. If φ is true in $\mathbb{N}$, then it is provable in Γ_n.

Proof. (Again, throughout the proofs formulas are understood to be arithmetical.) As to (a), it is enough to show that if φ is a Δ_0 sentence then

$$\mathbb{N} \models \varphi \Rightarrow \Gamma_n \vdash \varphi. \tag{2}$$

Indeed, the right-to-left implication in (a) follows straight from this: if $\mathbb{N} \not\models \varphi$ then $\mathbb{N} \models \neg\varphi$, but on the other hand $\neg\varphi$ is Δ_0, and thus $\Gamma_n \vdash \neg\varphi$. The proof of (2) is by induction on the complexity of $\varphi \in \Delta_0$.

One starts with the case when φ is atomic, i.e. φ is of the form $t = s$, or $t < s$: it may be convenient here to show first by induction on $m \in \mathbb{N}$, that $\mathbb{N} \models t = m \Rightarrow \Gamma_n \vdash t = m$ (the case $m = 0$ is worked out by additional induction on the complexity of t; the inductive step is easily dealt with, by going from $\mathbb{N} \models t = m + 1$ to $\mathbb{N} \models P(t) = m$, where P is the predecessor function); the general case when φ is $t = s$ easily reduces to this case, by taking m to be the number such that $m = s^{\mathbb{N}}$, hence $\mathbb{N} \models s = m$, and $\mathbb{N} \models t = m$. A similar argument works when φ is of the form $t < s$.

One of the inductive steps in the induction on the complexity of φ is when $\varphi =: (\forall v \leq t)\psi(v)$, where t is a term, ψ is Δ_0, and $\mathbb{N} \models \varphi$. By induction assume that for every closed term s, if $\mathbb{N} \models \psi(s)$ then $\Gamma_n \vdash \psi(s)$. Let $t^{\mathbb{N}} = m$. It follows by (1), that $\Gamma_n \vdash \psi(0) \wedge \cdots \wedge \psi(m)$. Moreover, $\Gamma_n \vdash t = m$: thus $\Gamma_n \vdash (\forall v \leq t)\psi(v)$.

We skip the rest of the proof, which follows the familiar lines of similar proofs, relative to formal systems in which one can represent the primitive recursive relations, see e.g. [13, 14].

As to (b), let $\exists v_1 \ldots \exists v_k \psi(v_1, \ldots, v_k)$ be a Σ_1 sentence, with ψ in Δ_0, and assume that $\mathbb{N} \models \exists v_1 \ldots \exists v_k \psi(v_1, \ldots, v_k)$: then there exist natural numbers $m_1, \ldots, m_k$ such that $\mathbb{N} \models \psi(m_1, \ldots, m_k)$, but $\psi(m_1, \ldots, m_k)$ is a true Δ_0 sentence, and thus $\Gamma_n \vdash \psi(m_1, \ldots, m_k)$, hence we can conclude that $\Gamma_n \vdash \exists v_1 \ldots \exists v_k \psi(v_1, \ldots, v_k)$. □

In the sequel, $T(e, x, y)$ denotes Kleene's predicate:

> *y codes a terminating computation of the partial computable function φ_e, coded by e, on input x,*

and $U(y)$ denotes the final value of the computation coded by y, that is, $U(y) = \varphi_e(x)$. Then

Lemma 3.5. *The relation $F(e, x, z)$:*

$$z = U(\min\{y : T(e, x, y)\}$$

is definable in $\mathbb{N}$ by a $\Sigma_1 n$ arithmetical formula $\Phi(v_0, v_1, v_2)$, for which Γ_n proves the uniqueness of v_2, that is

$$\Gamma_n \vdash \Phi(v_0, v_1, v_2) \wedge \Phi(v_0, v_1, v_2') \rightarrow v_2 = v_2'.$$

Proof. The relation $F(e, x, z)$ is r.e., and thus by the Matyasevich-Davis-Putnam-Robinson Theorem, see [15], there exists a quantifier-free polynomial relation $P_F(e, x, z, y_1, \ldots, y_m)$ (thus an arithmetical formula) such that

$$\mathbb{N} \models F(e, x, z) \Leftrightarrow \exists y_1 \cdots \exists y_m P_F(e, x, z, y_1, \ldots, y_m).$$

Let $G(v_0, v_1, v_2, v)$ be the Δ_0 formula

$$(\exists w_1 \leq v) \ldots (\exists w_m \leq v)\, P_F(v_0, v_1, v_2, w_1, \ldots, w_m).$$

Then the Σ_1 formula $\exists v\, G(v_0, v_1, v_2, v)$ defines $F(e, x, z)$, i.e.,

$$\mathbb{N} \models F(e, x, z) \Leftrightarrow \Gamma_n \vdash \exists v G(e, x, z, v).$$

By a familiar trick (see for instance [14, Selection Theorem 7.11]), from any such formula $\exists v\, G(v_0, v_1, v_2, v)$ defining $F(e, x, z)$, one can build a (still Σ_1) formula $\Phi(v_0, v_1, v_2)$ still defining $F(e, x, z)$, and proving uniqueness of v_2 as desired. □

Theorem 3.6. *Let $\mathbb{M}$ be a model of Γ_n, and define for all $e, x \in \mathbb{M}$,*

$$\varphi_e^{\mathbb{M}}(x) = \begin{cases} \text{the unique } z \in \mathbb{M} \text{ such that } \mathbb{M} \models \Phi(e, x, z), & \text{if such a } z \text{ exists} \\ \text{①}, & \text{otherwise} \end{cases}$$

Then for all finite e, x, z,

$$\varphi_e(x) = z \Rightarrow \varphi_e^{\mathbb{M}}(x) = z.$$

In other words, the computable functions can be made total inside $\mathbb{M}$.

Proof. (We identify finite numbers in $\mathbb{M}$ with their copies in $\mathbb{N}$.) If e, x, z are finite, and $\varphi_e(x) = z$ then $\mathbb{N} \models \Phi(e, x, z)$, thus by Lemma 3.1, $\Gamma_n \vdash \Phi(e, x, z)$, which implies that $\mathbb{M} \models \Phi(e, x, z)$, and finally $\varphi_e^{\mathbb{M}}(x) = z$. □

Remark 3.7. It is quite possible that in a model $\mathbb{M}$ of Γ_n there are finite numbers e, n, m such that $\varphi_e(n)$ is undefined but $\mathbb{M} \models \Phi(e, n, m)$.

Theorem 3.8. *For every $n > 0$, Γ_n is undecidable.*

Proof. We show in fact that Γ_n is creative. Let (A_0, A_1) be a pair of effectively inseparable sets, see e.g. [16], and by Lemma 3.1 let $\exists v \psi_0(u, v)$ and $\exists v \psi_1(u, v)$ be Σ_1 formulas such that, for $i = 0, 1$, and for all $x \in \mathbb{N}$,

$$x \in A_i \Leftrightarrow \mathbb{N} \models \exists v \psi_i(x, v).$$

For $i = 0, 1$, let

$$B_i = \{x : \Gamma_n \vdash \exists v(\psi_i(x, v) \wedge (\forall v' \leq v)\neg\psi_{1-i}(x, v'))\}.$$

It is easy to see that $A_i \subseteq B_i$: indeed, if $x \in A_i$, then there exists m such that $\mathbb{N} \models \psi_i(x, m)$, but then

$$\Gamma_n \vdash \varphi_i(x, m) \wedge (\forall v' \leq m)\neg\varphi_{1-i}(x, v'),$$

using (1) and the fact that A_0 and A_1 are disjoint. Moreover, B_0 and B_1 are disjoint: if, for instance, x is a finite natural number such that

$$\Gamma_n \vdash \exists v(\psi_0(x, v) \wedge (\forall v' \leq v)\neg\psi_1(x, v')) \wedge \exists w\,(\psi_1(x, w) \wedge (\forall w' \leq w)\neg\psi_0(x, w'))$$

then choose (using e.g. Rule C, see [13]) such v, w, but from the order axioms of Γ_n we have that $\Gamma_n \vdash v \leq w \vee w \leq v$, which contradicts consistency of Γ_n.

Therefore we can conclude that the pair (B_0, B_1) is effectively inseparable as well, being an r.e. disjoint pair of supersets of A_0 and A_1. Since each half of an effectively inseparable pair is a creative set, it follows for instance that B_0 is creative, but, by its very definition, B_0 is many-one reducible to the set of theorems of Γ_n, therefore showing that the set of theorems of Γ_n is creative. □

4. Consistency of Γ_n

In this section we prove that Γ_n is interpretable in $I\Delta_0 + \Omega_1$, and hence its consistency is no more problematic than the consistency of predicative arithmetic.

Let $\mathbb{M}$ be a nonstandard model of $I\Delta_0 + \Omega_1$, and from $\mathbb{M}$ and an infinite element N of $\mathbb{M}$, let us build a structure $\mathbb{K}$, with corresponding ①, and ∞_{-n}, $n = 1, 2, \ldots$, as explained at the beginning of Section 2; in particular, for every function symbol f, we take $f^{\mathbb{K}}$ to be the truncation of f to ①.

Theorem 4.1. *All axioms of Γ_n are satisfied in $\mathbb{K}$. Hence, for every n, the consistency of Γ_n is implied by the consistency of $I\Delta_0 + \Omega_1$.*

Proof. Most of the axioms of Γ_n are easy to verify. We have already discussed at the beginning of Section 2 the validity of the axioms for truncations of the exponential function or of the factorial function. We verify that full induction holds in $\mathbb{K}$. To this purpose, it suffices to verify the Least Number Principle. That is, given a formula $\psi(x_1, \ldots, x_n, u)$ and $a_1, \ldots, a_n \in \mathbb{K} = [0, ①]$, if there is a $b \in \mathbb{K}$ such that $\psi(a_1, \ldots, a_n, b)$ holds in $\mathbb{K}$, then there is a minimum b with this property. To this purpose, from $\psi(x_1, \ldots, x_n, u)$ build a formula $\psi'(x_1, \ldots, x_n, u, z)$ as done in the proof of Theorem 2.5. Notice that ψ' is a Δ_0 formula, since all its quantifiers are bounded. An easy computation shows that

$$\mathbb{K} \models \psi(a_1, \ldots, a_n, b) \Leftrightarrow \mathbb{M} \models \psi'(a_1, \ldots, a_n, b, ①).$$

Since $\mathbb{M}$ is a model of Δ_0 induction, there is a minimum $b_0 \leq ①$ in $\mathbb{M}$ such that $\mathbb{M} \models \psi'(a_1, \ldots, a_n, b_0)$. But then b_0 is the minimum b such that $\mathbb{K} \models \psi(a_1, \ldots, a_n, b)$. □

Problem 4.2. We have seen that given any model $\mathbb{M}$ of $I\Delta_0 + \Omega_1$ there is a model $\mathbb{K}$ of Γ_n which is obtained restricting the universe of $\mathbb{M}$ to the interval $[0, ①]$ for some element ① of $\mathbb{M}$. We will denote it by $\mathbb{M}[0, ①]$. Is it true that given any model $\mathbb{K}$ of Γ_n there is a model $\mathbb{M}$ of $I\Delta_0 + \Omega_1$ such that $\mathbb{K} = \mathbb{M}[0, ①]$, for some $① \in \mathbb{M}$?

Problem 4.3. The consistency of the whole theory Γ follows from the consistency of all Γ_n using the Compactness Theorem, see e.g. [13]. However, we do not know if this theorem can be formalized in predicative arithmetic, and hence we do not know if the consistency of Γ can be proved inside $I\Delta_0 + \Omega_1$.

4.1. **Gödel's Second Incompleteness Theorem and the theory Γ_n.** Gödel's Second Incompleteness Theorem says that a sufficiently strong computably axiomatizable and consistent theory cannot prove its own consistency. In this subsection we provide a semantic argument which might be interpreted as a failure of the theorem in some consistent extensions of Γ_n. (In any case, this does not yield a contradiction, because Γ_n is not a system for arithmetic in the usual sense. Moreover, to get the failure of Gödel's Second Incompleteness Theorem we would have to formalize our argument inside Γ_n). Let $\mathrm{Proof}(x, y)$ be a Δ_0 predicate expressing

$$x < ① \textit{ and } x \textit{ codes a proof in } \Gamma_n \textit{ of the formula coded by } y,$$

and let us express consistency of Γ_n as $\mathrm{Con}_{\Gamma_n} =: \neg\exists x(x < ① \wedge \mathrm{Proof}(x, \perp))$. Then:

Theorem 4.4. *From any model $\mathbb{M}$ of Γ_n one can obtain a model of $\Gamma_n \cup \{\mathrm{Con}_{\Gamma_n}\}$. Hence, the consistency of Γ_n implies the consistency of $\Gamma_n \cup \{\mathrm{Con}_{\Gamma_n}\}$.*

Proof. If Con_{Γ_n} holds in $\mathbb{M}$, we are done. Otherwise, using the Least Number Principle we can see that there is a minimum $y_0 < ①$ such that $\mathbb{M} \models \mathrm{Proof}(y_0, \perp)$. Clearly, y_0 is infinite, otherwise $\mathbb{M}$ in this case would verify a Δ_0 arithmetical sentence ψ expressing that y_0 codes a proof of $\perp$, but then by Lemma 3.4 we would have $\mathbb{N} \models \mathrm{Proof}(y_0, \perp)$, and $\Gamma_n \vdash \perp$, thus Γ_n would have no model. Let $N < y_0$ be an infinite element of $\mathbb{M}$, and let us perform the same construction as at the beginning of Section 2, that is, consider the logarithm of an infinite N iterated $2n$ times, call it ∞'_{-n}, and let for $k < n-1$, $\infty'_{-n+k+1} = (\infty'_{-n+k}!)^{\infty'_{-n+k}}$ and $①' = (\infty'_{-1}!)^{\infty'_{-1}}$. Then $\mathbb{M}[0, ①']$ with all operations truncated to $①'$, is a model of Γ_n in which there is no proof of $\perp$, and hence it is a model of $\Gamma_n \cup \{\mathrm{Con}_{\Gamma_n}\}$, as desired. □

5. Towards a treatment of mathematics inside Γ_n.

In this section we introduce some mathematics in Γ_n. Most of our results are imitations of results in the usual theory of grossone. But we must recall that in the classical theory of grossone, one may use numbers that are bigger than the grossone, while in Γ_n every number is smaller than it: this is illustrated in Figure 1. Moreover, we want to use only extended natural numbers and not other numbers like rational numbers or real numbers. In order to make up for these limitations, we start with a "copy" of the natural numbers of the form $[0, \infty_{-n}]$, with n sufficiently large, and then we use the numbers greater than ∞_{-n} to form the wider class of the extended natural numbers. This will allow us, for instance, to code subsets of $[0, \infty_{-n}]$, sequences of subsets, etc. Moreover, we will see that we can represent the integers and the rational numbers by extended natural numbers in $[0, \infty_{-n+1}]$.

In the rest of the section we mostly argue informally, as if we were working in some, not specified, model of Γ_n. Of course, the results and the properties that we point out derive from formal theorems of Γ_n, of which, when not completely evident, we hint at the proofs.

5.1. **The integers and the rational numbers with denominator ∞_{-n}.** For many purposes, it is sufficient to restrict ourselves to rational numbers with denominator ∞_{-n}, for a suitable n (it is convenient to take $n > 1$) and with numerator an integer between $-\infty_{-n}^2$ and $+\infty_{-n}^2$. This restriction is justified by the fact that in a nonstandard model of the reals, if N is an infinite natural number then every standard real number α can be approximated up to the infinitesimal $\frac{1}{N}$ by a rational of the form $\frac{a}{N}$, for some integer a with $-N^2 \leq a \leq N^2$ (to see this, let a be the maximum such that $\frac{a}{N} \leq \alpha$). In our case, taking $N = \infty_{-n}$, this leads us to represent the rational numbers in the form

$$\frac{a}{\infty_{-n}}, \quad \text{with } -\infty_{-n}^2 \leq a \leq \infty_{-n}^2.$$

The set of these rational numbers will be denoted by $\mathbb{Q}_n$. Now the denominator of these numbers is fixed, and hence we can code these rational numbers by their numerator a, which is an integer between $-\infty_{-n}^2$ and ∞_{-n}^2. Let us call $\mathbb{Z}_n$ the set of these integers.

We can represent each integer in $\mathbb{Z}_n$ as a number in $[0, 2 \cdot \infty_{-n}^2]$, and hence, in $[0, \infty_{-n+1}]$: this identification can be made, for instance, through the coding $^* : \mathbb{Z}_n \longrightarrow [0, 2 \cdot \infty_{-n}^2]$,

$$a^* = \begin{cases} 0, & \text{if } a = 0, \\ 2 \cdot a - 1, & \text{if } a > 0, \\ 2 \cdot b, & \text{if } a = -b, \text{ with } 0 < b \leq \infty_{-n}^2. \end{cases}$$

Likewise, we may define, for all $q = \frac{a}{\infty_{-n}} \in \mathbb{Q}_n$ (where $a \in \mathbb{Z}_n$),

$$q^\circ = \left(\frac{a}{\infty_{-n}}\right)^\circ = a^*,$$

thus coding every rational number in $\mathbb{Q}_n$ by an extended natural number. Conversely, every natural number $m \in [0, 2 \cdot \infty_{-n}^2]$ is the code of both an integer $^*m$, defined by

$$^*m = \begin{cases} 0 & \text{if } m = 0, \\ -k & \text{if } m = 2 \cdot k \text{ and } m \neq 0, \\ k & \text{if } m = 2 \cdot k - 1, \end{cases}$$

and of a rational $^\circ m \in \mathbb{Q}_n$ defined by $^\circ m = \frac{^*m}{\infty_{-n}}$.

Using these codes, we can represent any k-ary function f from $(\mathbb{Q}_n)^k$ into $\mathbb{Q}_n$ by means of the function f° from $([0, 2 \cdot \infty_{-n}^2])^k$ into $[0, 2 \cdot \infty_{-n}^2]$, defined by

$$f^\circ(m_1, \ldots, m_k) = (f(^\circ m_1, \ldots, ^\circ m_k))^\circ.$$

Clearly, when f° is (the restriction of) a polynomial time function, f can be represented in the theory Γ_n.

Remark 5.1. Rational numbers in $\mathbb{Q}_n$ are not closed under some very natural operations, for instance, they are not closed under product. However, in many significant cases, we may consider such operations as maps from $\mathbb{Q}_n$ into $\mathbb{Q}_{n-1}$. For instance, product can be considered as a map from $\mathbb{Q}_n^2$ into $\mathbb{Q}_{n-1}$, defined by

$$\frac{a}{\infty_{-n}} \cdot \frac{b}{\infty_{-n}} = \frac{a \cdot b \cdot q}{\infty_{-n+1}},$$

where q is the quotient of ∞_{-n+1} and ∞_{-n}^2: recall that ∞_{-n}^2 divides ∞_{-n+1}, by Lemma 2.2. Note that $\mathbb{Q}_{n-1}$ is in turn not closed under product, but we may restrict ourselves to the collection $\mathbb{Q}_{n-1}^*$ of all elements $\frac{q}{\infty_{-n+1}}$ such that there are a finite number k and elements $x, y \in [-\infty_{-n}^k, \infty_{-n}^k]$, with at least one of x, y different from 0, such that $x \cdot \infty_{-n+1} = q \cdot y$. The following fact is readily seen:

Fact 5.2. $\mathbb{Q}_{n-1}^*$ *forms a field.*

Proof. We prove that $\mathbb{Q}_{n-1}^*$ is closed under products. So assume that $\frac{q}{\infty_{-n+1}}, \frac{q'}{\infty_{-n+1}} \in \mathbb{Q}_{n-1}^*$, i.e., there are finite numbers k, k', and there exist $x, y \in [-\infty_{-n}^k, \infty_{-n}^k]$, and $x', y' \in [-\infty_{-n}^{k'}, \infty_{-n}^{k'}]$, such that $x \cdot \infty_{-n+1} = q \cdot y$ and $x' \cdot \infty_{-n+1} = q' \cdot y'$. Thus,

$$x \cdot x', y \cdot y' \in [-\infty_{-n}^{k+k'}, \infty_{-n}^{k+k'}].$$

It follows from Lemma 2.2 that, for every finite m,

$$u \leq \infty_{-n}^m \to u | \infty_{-n+1},$$

and thus we have that $\infty_{-n+1} = y \cdot y' \cdot \hat{q}$, for some $\hat{q}$. Thus,

$$\begin{aligned} \frac{q \cdot q'}{\infty_{-n+1}^2} &= \frac{x \cdot x'}{y \cdot y'} \\ &= \frac{x \cdot x' \cdot \hat{q}}{y \cdot y' \cdot \hat{q}} = \frac{x \cdot x' \cdot \hat{q}}{\infty_{-n+1}}. \end{aligned}$$

Then the desired claim follows from the following fact:

$$(x \cdot x') \cdot \infty_{-n+1} = (x \cdot x' \cdot \hat{q}) \cdot (y \cdot y').$$

Next, we prove that $\mathbb{Q}_{n-1}^*$ is closed under inverse. Let $\frac{q}{\infty_{-n+1}} \in \mathbb{Q}_{n-1}^*$, with $q \neq 0$, and let k and $x, y \in [-\infty_{-n}^k, \infty_{-n}^k]$ (with not both of them equal to 0) be such that $x \cdot \infty_{-n+1} = q \cdot y$: then $x, y \neq 0$. Hence $\frac{q}{\infty_{-n+1}} = \frac{x}{y}$, and thus $\frac{\infty_{-n+1}}{q} = \frac{y}{x}$. But by Lemma 2.2, x divides ∞_{-n+1}, and thus $\infty_{-n+1} = q' \cdot x$ for some q'. It follows

$$\frac{\infty_{-n+1}}{q} = \frac{y \cdot q'}{x \cdot q'} = \frac{y \cdot q'}{\infty_{-n+1}},$$

and $y \cdot \infty_{-n+1} = (y \cdot q') \cdot x$, from which the claim easily follows. We leave to the reader to verify the remaining cases that need to be checked to show that $\mathbb{Q}_n$ is a field. □

Unfortunately, such a collection $\mathbb{Q}_{n-1}^*$ is not a set in our sense, as is not first order definable.

5.2. **Sets and cardinalities: proving that the part is less than the whole.** In this section we show how to treat the issue of cardinalities of sets, reobtaining in our context some of the most characterizing results of the theory of grossone. Other authors have provided already rigorous approaches to the issue of cardinality in the theory of grossone, see e.g. [17, 18]. The novelty of our approach lies basically in the fact that we work in a very weak first order formal system (compared, for instance, to [17], where a second order system extending Peano Arithmetic is employed).

In order to deal with sets and cardinalities, we will limit ourselves to *bounded* sets (i.e., contained in $[0, \infty_{-n}]$, for a chosen $n > 1$) which are definable by a formula. In fact, by way of exemplification, we will restrict to subsets of $[0, \infty_{-2}]$. The characteristic function C_φ of a subset X of $[0, \infty_{-2}]$ definable by a formula φ, is a term of the language, as shown in the following theorem.

Theorem 5.3. *For every formula φ, defining a subset of $[0, \infty_{-2}]$, we have that C_φ is a term of Γ_2, and hence, a term of Γ_n for all $n \geq 2$.*

Proof. First of all, note that $\max\{x, y\}$ and $\min\{x, y\}$ are definable in our logic by $\max\{x, y\} = x + (y \ominus x)$ and $\min\{x, y\} = x \ominus (x \ominus y)$. We now prove the claim by induction on the formula φ which defines X. If φ has the form $t_1 = t_2$, then $C_\varphi = (1 \ominus (t_1 \ominus t_2)) \ominus (t_2 \ominus t_1)$; if φ has the form $t_1 \leq t_2$, then $C_\varphi = 1 \ominus (t_1 \ominus t_2)$. For the induction steps, let $C_{\neg\varphi} = 1 \ominus C_\varphi$, $C_{\varphi\wedge\psi} = \min\{C_\varphi, C_\psi\}$, $C_{\varphi\vee\psi} = \max\{C_\varphi, C_\psi\}$, $C_{\exists u\varphi}(x) = \min\{1, \Sigma_{C_\varphi}(x, i, ①)\}$, and $C_{\forall u\varphi} = 1 \ominus C_{\exists u\neg\varphi}$. □

Remark 5.4. For the ease of the reader, and to help intuition, we will use the usual notations for sums, i.e. $\sum_{i=0}^{\infty_{-2}} C_\varphi(i)$ represents $\Sigma_{C_\varphi}(i, \infty_{-2}, ①)$ (which is clearly equal to $\Sigma_{C_\varphi}(i, \infty_2, \infty_{-1})$ as well): recall that for every term t, the language is equipped with a function symbol Σ_t, to be interpreted as sums over values taken by t. Moreover, as usual, it is understood that $2^i =: \mathrm{Exp}_{|①|}(2, i)$.

Definition 5.5. Let C_X be the characteristic function of a definable $X \subseteq [0, \infty_{-2}]$.

(1) We *code* X by the number $X^\#$ in $[0, \infty_{-1}]$, namely,

$$X^\# = \sum_{i=0}^{\infty_{-2}} C_X(i) \cdot 2^i;$$

(2) The *cardinality* of X, is

$$\mathrm{Card}(X) = \sum_{i=0}^{\infty_{-2}} C_X(i).$$

Note that the above sums exist because $C_X(i) \cdot 2^i$ and $C_X(i)$ are terms, and these sums are $\leq \infty_{-1}$, as follows from (1) and (2) of the following lemma.

Lemma 5.6. *For definable subsets $X, Y \subseteq [0, \infty_{-2}]$, we have*

(1) $[0, \infty_{-2}]^\# = P(2^{\infty_{-2}+1}) < \infty_{-1}$ (recall that P is function symbol representing the predecessor);

(2) $X \subseteq Y \to X^{\#} \leq Y^{\#}$*;*
(3) $X \neq Y \to X^{\#} \neq Y^{\#}$*;*
(4) $x \in X \to x \leq |X^{\#}|$*;*
(5) for all $x \in [0, \infty_{-2}]$ *one has:* $x \in X$ *if and only if*

$$\exists z < 2^x \exists v \leq X^{\#}(X^{\#} = z + 2^x + v \cdot 2^{x+1}).$$

Proof. To show (1), Γ_2 proves by induction on x that

$$x \leq |①| \to \sum_{i=0}^{x} 2^i = P(2^{x+1}),$$

so the result follows by taking $x = \infty_{-2}$. To show (2), one can use the following fact that can be easily seen to be proved by Γ_2 by induction:

$$y \leq \infty_{-2} \wedge ((\forall i \leq y)\, C_X(i) \leq C_Y(i)) \to \sum_{i=0}^{y} C_X(i) \cdot 2^i \leq \sum_{i=0}^{y} C_Y(i) \cdot 2^i.$$

This yields the claim by taking $y = \infty_{-2}$. We sketch the proof of (3): the argument can be easily turned into a formal proof in Γ_2. Let i_0 be the least such that $C_X(i_0) \neq C_Y(i_0)$: assume for definiteness that $C_X(i_0) = 1$. If $X^{\#} = Y^{\#}$ then

$$\sum_{j=0}^{\infty_{-2} \ominus i_0} C_X(i_0 + j) \cdot 2^{i_0+j} = \sum_{j=0}^{\infty_{-2} \ominus i_0} C_Y(i_0 + j) \cdot 2^{i_0+j} :$$

both numbers are divisible by 2^{i_0}, but in the former case the quotient is odd whereas in the latter case the quotient is even, thus giving a contradiction.

We skip the proofs of the remaining claims. □

Remark 5.7. Building on item (5) of the previous lemma, we will write

$$xEy =: \exists z < 2^x \exists v \leq y(y = z + 2^x + v \cdot 2^{x+1}).$$

Thus, in view of the previous lemma, bounded definable sets can be identified with numbers $\leq \infty_{-1}$ (in fact $\leq P(2^{\infty_{-2}+1})$): any $y \leq \infty_{-1}$ "corresponds" to the set $Y = \{x \leq \infty_{-2} : xEy\}$.

Likewise, we may code a function f as the set of codes of pairs $\langle x, y\rangle$ such that $y = f(x)$. Note that if the domain of f is a closed interval $[0, x]$ with $x \leq \infty_{-2}$, then we can express $f(i)$ for all $i \leq x$ as a term of Γ_n,

$$f(i) = \mu z \leq |f^{\#}|.\langle i, z\rangle E f^{\#},$$

where $f^{\#}$ is the code of f. Given two definable subsets X, Y of $[0, \infty_{-2}]$ and a number $h \in [0, \infty_{-1}]$, we can express the notion h *codes a function from* X *to* Y as the conjunction of the following formulas:

$$(\forall x \leq \infty_{-2})(xEh \to (\pi_1(x)EX^{\#} \wedge \pi_2(x)EY^{\#})$$
$$(\forall x \leq \infty_{-2})(xEX^{\#} \to \exists y \leq \infty_{-2}(yEY^{\#} \wedge \langle x, y\rangle\, Eh))$$
$$(\forall x \leq \infty_{-2})(\forall y \leq \infty_{-2})(\forall z \leq \infty_{-2})\left((\langle x, y\rangle\, Eh \wedge \langle x, z\rangle\, Eh) \to y = z\right).$$

It is easy to express concepts like h *codes an injective* (respectively, *surjective, bijective*) *function from* X *to* Y, using again bounded formulas. In particular we say that there is a *codable bijection* of X onto Y, if there exists a code of a bijection of X onto Y.

In Γ_2 we can reobtain some theorems of the theory of grossone concerning cardinalities:

Theorem 5.8. *Let* X, Y *be definable subsets of* $[0, \infty_{-2}]$. *Then:*

(1) $\mathrm{Card}(X) = \mathrm{Card}(Y)$ *if and only if there is a codable bijection between* X *and* Y*;*
(2) there is no codable bijection between a set X *and a proper subset of it.*

Proof. We sketch how to prove the two claims:

(1) Suppose $\mathrm{Card}(X) = \mathrm{Card}(Y)$. We prove that there is a code of a bijection from X into Y by induction on $k = \mathrm{Card}(X) = \mathrm{Card}(Y)$, applying induction to the formula

$$\forall \text{ definable } X, Y \subseteq [0, \infty_{-2}]\, (\mathrm{Card}(X) = \mathrm{Card}(Y) = k \to k \text{ codes a bijection from } X \text{ onto } Y):$$

contrary to appearance, this is in fact a formula of our language, as by Remark 5.7 the two seemingly second order quantifiers $\forall X$ and $\forall Y$ correspond to quantifiers on variables $\leq \infty_{-1}$, and "codes a bijection from X onto Y" is first order too. The claim is trivial if $\mathrm{Card}(X) = \mathrm{Card}(Y) = 0$. Suppose the claim holds for $\mathrm{Card}(X) = \mathrm{Card}(Y) = k$ and let us prove it for $\mathrm{Card}(X) = \mathrm{Card}(Y) = k + 1$. Let z be maximal such that 2^z divides $X^\#$ and let w be maximal such that 2^w divides $Y^\#$. Then $X^\# - 2^z$ and $Y^\# - 2^w$ code $X' = X \setminus \{z\}$ and $Y' = Y \setminus \{w\}$, respectively. Clearly, $\mathrm{Card}(X') = \mathrm{Card}(Y') = k$ and by the induction hypothesis there is a code $f^\#$ of a bijection f from X' onto Y'. Then $f^\# + 2^{\langle z,w\rangle}$ codes a bijection from X onto Y.

Conversely, we prove by induction on $\mathrm{Card}(X)$ that if there is a code of a bijection from X onto Y, then $\mathrm{Card}(X) = \mathrm{Card}(Y)$: the argument which we are going to sketch can be easily formalized in Γ_2. If $\mathrm{Card}(X) = 0$, the claim is trivial. Suppose the claim holds for $\mathrm{Card}(X) = k$, and let us prove it for $\mathrm{Card}(X) = k + 1$. Let f^* be the code of a bijection from X onto Y, let z be maximal such that 2^z divides $X^\#$ and let w be the unique element of Y such that $\langle z, w\rangle\, E f^\#$. Then $f^\# - 2^{\langle z,w\rangle}$ codes a bijection from $X' = X \setminus \{z\}$ onto $Y' = Y \setminus \{w\}$ and hence, by the induction hypothesis, $\mathrm{Card}(X') = \mathrm{Card}(Y') = k$. It follows that $\mathrm{Card}(X) = \mathrm{Card}(Y) = k + 1$.

(2) If X is a proper subset of Y, then for all $i \leq \infty_{-2}$, $C_X(i) \leq C_Y(i)$, and for some least y_0, $C_X(y_0) < C_Y(y_0)$. But then by induction on y we can see that

$$(y < y_0 \to \sum_{i=0}^{y} C_X(i) = \sum_{i=0}^{y} C_Y(i)) \wedge (y_0 \leq y \to \sum_{i=0}^{y} C_X(i) < \sum_{i=0}^{y} C_Y(i)).$$

Letting $y = \infty_{-2}$ we get $\mathrm{Card}(X) < \mathrm{Card}(Y)$. Hence, by the previous item, there is no codable bijection from X onto Y.

□

Remark 5.9. In the previous theorem the assumption that bijections are coded in the theory is essential. For instance, if α is an infinite natural number, the map $0 \mapsto \alpha + 1$, $n + 1 \mapsto n$, with n finite, and $\beta \mapsto \beta$ for $\beta \leq \alpha$, with β infinite, is a bijection between $[0, \alpha]$ and $[0, \alpha + 1]$.

The following corollary corresponds to Sergeyev's Divisibility Axiom, according to which (see e.g. [1]), the cardinality of the set of even numbers is $\frac{①}{2}$; we regain this result in our context, via the correspondence of Sergeyev's ① with our ∞_{-2}. (Notice that, in order to get the same estimates as Sergeyev, the presence of 0 forces us, in the following corollary, to take the natural numbers to be $[0, \infty_2)$, and not $[0, \infty_2]$ as done so far.)

Corollary 5.10. *Let $\mathcal{N} = [0, \infty_{-2})$ be the set of natural numbers, and let $\mathrm{Ev} = \{x < \infty_{-2} : \exists y \, x = y \cdot 2\}$ be the (definable) set of even numbers. Then*

$$\begin{aligned} \mathrm{Card}(\mathcal{N}) &= \infty_{-2} \\ \mathrm{Card}(\mathrm{Ev}) &= \frac{\infty_{-2}}{2}. \end{aligned}$$

Proof. By induction on x, one shows that the following hold:

$$x \leq P(\infty_{-2}) \to \sum_{i=0}^{x} C_{\mathcal{N}}(i) = x + 1$$

$$x \leq P(\infty_{-2}) \wedge C_{\mathrm{Ev}}(x) = 1 \to \sum_{i=0}^{x} C_{\mathrm{Ev}}(i) = \frac{x+1}{2}.$$

□

Notice that the set of odd numbers Od has $\mathrm{Card}(\mathrm{Od}) = \frac{\infty_{-2}}{2}$, as well.

5.3. **Elementary measure theory.** In Γ_n with $n \geq 2$, we can also develop a very elementary measure theory for subsets of $\mathbb{Q}_n$. To this purpose, we observe that the unit interval

$$[0, 1) = \left[\frac{0}{\infty_{-n}}, \frac{\infty_{-n}}{\infty_{-n}}\right)$$

contains ∞_{-n} elements, and since it is customary to assign measure 1 to $[0, 1)$, we will define, for any definable subset X of $[-\infty_{-n}, \infty_{-n}]$ of rational numbers ($-\infty_{-n}$ and ∞_{-n} come by simplifying $\frac{-\infty_{-n}^2}{\infty_{-n}}$ and $\frac{\infty_{-n}^2}{\infty_{-n}}$),

$$\mu(X) = \frac{\mathrm{Card}(X)}{\infty_{-n}} :$$

notice that as a set of rationals, X corresponds in fact to a set of integers, and thus to a subset of $[0, 2 \cdot \infty_{-n}^2]$, thus $\mathrm{Card}(X) \leq \infty_{-n+1}$.

It is not difficult to see, for definable subsets X, Y of rationals in $[-\infty_{-n}, \infty_{-n}]$ that $\mathrm{Card}(X \cup Y) \leq \mathrm{Card}(X) + \mathrm{Card}(Y)$, and if $X \cap Y = \emptyset$ then $\mathrm{Card}(X \cup Y) = \mathrm{Card}(X) + \mathrm{Card}(Y)$. Therefore μ is an additive measure. In fact, if $n > 2$, then we can code *bounded* sequences (i.e. sequences, whose indices lie in a bounded set) of uniformly definable subsets of $[-\infty_{-n}, \infty_{-n}]$ in Γ_n and we obtain:

Theorem 5.11. *μ is additive, that is, if $b \leq \infty_{-n}$ and $\{X_a : a \leq b\}$ is a sequence of uniformly definable pairwise disjoint subsets of rationals in $[-\infty_{-n}, \infty_{-n}]$, then*

$$\mu(\bigcup_{a \leq b} X_a) = \sum_{a \leq b} \mu(X_a).$$

Proof. To see this, from a formula $\varphi(x, a)$ that uniformly defines the sequence of sets for $a \leq b$, one builds suitable terms $c(y)$ and $c(a)$ expressing $\mathrm{Card}(\bigcup_{a \leq y} X_a)$ and $\mathrm{Card}(X_a)$, respectively, and then, using the assumptions, one checks by induction on y that

$$y \leq b \rightarrow \mathrm{Card}(\bigcup_{a=0}^{y} X_a) = \sum_{a=0}^{y} \mathrm{Card}(X_a).$$

□

Note that the measure μ just introduced has the following additional properties which immediately follow from properties of cardinality:

(1) if X is a proper subset of Y, then $\mu(X) < \mu(Y)$; in particular, every non-empty set has positive measure;
(2) Any two subsets of $\mathbb{Q}_n$ with the same cardinality have the same measure.

For different approaches toward the issue of cardinality vs. measures, see also [17, 18].

5.4. **Series.** In this section we give a look at series of rational numbers. We refer the reader to the interesting papers [4, 19] for a thorough account on how series are treated within the theory of grossone. By abuse of language we use for series of rational numbers the same notation used for series of natural numbers. Precisely, we will discuss series of the form $\sum_{i=0}^{\infty_{-n}} a_i$ such that $a_i \in \mathbb{Q}_n$ and the map $i \mapsto a_i^\circ$ is coded in Γ_n. In other words, we assume that if $a_i = \frac{b_i}{\infty_{-n}}$, then letting $f(i) = b_i^*$, there is a number $f^\# < \infty_{-n+1}$ such that (the quantifier bound ∞_{-n+1} in the following expression is quite generous: it is enough of course to bound $\langle x, y \rangle$ for $x, y \leq \infty_{-n}$)

$$(\forall x \leq \infty_{-n+1})(xEf^\# \rightarrow \pi_1(x) \leq \infty_{-n})$$
$$(\forall i \leq \infty_{-n})(\forall z \leq 2 \cdot \infty_{-n}^2)(\langle i, z \rangle Ef^\# \leftrightarrow z = f(i)).$$

Let us call *definable* a series for which there is a codable f as above.

Theorem 5.12. *Every definable series converges.*

Proof. Let $\sum_{i=0}^{\infty_{-n}} a_i$ be a definable series, and let f be a definable function such that $a_i = \frac{b_i}{\infty_{-n}}$, and $f(i) = b_i^*$. Then we can compute the code, $(\sum_{i=0}^{\infty_{-n}} a_i)^\circ$, of $\sum_{i=0}^{\infty_{-n}} a_i$, as follows. Let $C_{\mathrm{Ev}}(x)$ and $C_{\mathrm{Od}}(x)$ be the characteristic functions of the set of even numbers and of the set of odd numbers, respectively. Then (recall how negative integers are coded by even numbers, and positive integers are coded by odd numbers)

$$\left(\sum_{i=0}^{\infty_{-n}} a_i\right)^{\circ} = \left(2 \cdot \left(\sum_{i=0}^{\infty_{-n}} \frac{f(i)+1}{2} \cdot C_O(f(i)) \ominus \sum_{i=0}^{\infty_{-n}} \frac{f(i)}{2} \cdot C_E(f(i))\right) \ominus 1\right) + 2 \cdot \left(\sum_{i=0}^{\infty_{-n}} \frac{f(i)}{2} \cdot C_E(f(i)) \ominus \sum_{i=0}^{\infty_{-n}} \frac{f(i)+1}{2} \cdot C_O(f(i))\right).$$

Indeed, this codes the sum, as by induction on y one can prove :

$$y \leq \infty_n \to \left(\sum_{i=0}^{y} a_i\right)^{\circ} = \left(2 \cdot \left(\sum_{i=0}^{y} \frac{f(i)+1}{2} \cdot C_O(f(i)) \ominus \sum_{i=0}^{y} \frac{f(i)}{2} \cdot C_E(f(i))\right) \ominus 1\right) + 2 \cdot \left(\sum_{i=0}^{y} \frac{f(i)}{2} \cdot C_E(f(i)) \ominus \sum_{i=0}^{y} \frac{f(i)+1}{2} \cdot C_O(f(i))\right).$$

□

In order to simplify matters, we will identify the sum S of the series in $\mathbb{Q}_n$ with its code S° in $[0, \infty_{-n+1}]$.

We are now going to prove that the sum is independent of any codable rearrangements of the terms of the series.

Lemma 5.13. *For all $h < k \leq \infty_{-n}$,*

$$\sum_{i=0}^{k} a_i = \sum_{i=0}^{h} a_i + \sum_{i=0}^{k-h-1} a_{h+1+i}.$$

Proof. We sketch the proof which is by induction on $m = k - h - 1$, or more formally on the formula

$$\forall h \forall k \, (k \leq \infty_{-n} \wedge h < k \wedge k = m + h + 1 \to \sum_{i=0}^{k} a_i = \sum_{i=0}^{h} a_i + \sum_{i=0}^{m} a_{h+1+i}).$$

If $m = 0$, the claim is easy. Suppose the claim holds for $m = n$, and let us prove it for $m = n + 1$. By the induction hypothesis,

$$\sum_{i=0}^{k-1} a_i = \sum_{i=0}^{h} a_i + \sum_{i=0}^{k-h-2} a_{h+1+i},$$

hence, if $h < k - 1$,

$$\sum_{i=0}^{k} a_i = \sum_{i=0}^{k-1} a_i + a_k = \sum_{i=0}^{h} a_i + \sum_{i=0}^{k-h-2} a_{h+1+i} + a_k = \sum_{i=0}^{h} a_i + \sum_{i=0}^{k-h-1} a_{h+1+i}.$$

The remaining case $h = k - 1$ is trivial. □

Theorem 5.14. *For $h \leq \infty_{-n}$, if f is a permutation of $[0, h]$ coded in Γ_n, then* $\sum_{i=0}^{h} a_i = \sum_{i=0}^{h} a_{f(i)}$.

Proof. (Sketch) Since all denominators are equal to ∞_{-n}, we may assume that all a_i are integers. We reason by induction on h. The case $h = 0$ is trivial. For the induction step, let i_0 be such that $f(i_0) = h + 1$. We have $\sum_{i=0}^{h+1} a_i = \sum_{i=0}^{h} a_i + a_{h+1}$, while, by Lemma 5.13,

$$\sum_{i=0}^{h+1} a_{f(i)} = \sum_{i=0}^{P(i_0)} a_{f(i)} + \sum_{i=i_0+1}^{h+1} a_{f(i)} + a_{h+1},$$

where f codes a permutation of $[0, h + 1]$. Let $g(i) = f(i)$ if $i < i_0$ and $g(i) = f(i + 1)$ if $i \geq i_0$. Then g is a permutation of $[0, h]$. Note that g can be coded in Γ_n by

$$\sum_{i=0}^{i_0-1} 2^{\langle i,f(i)\rangle} + \sum_{i=0}^{h-i_0} 2^{\langle i_0+i,f(i_0+i+1)\rangle}.$$

By the induction hypothesis, $\sum_{i=0}^{h} a_{g(i)} = \sum_{i=0}^{h} a_i$, and

$$\begin{aligned}\sum_{i=0}^{h+1} a_{f(i)} &= \sum_{i=0}^{h} a_{g(i)} + a_{h+1} \\ &= \sum_{i=0}^{h} a_i + a_{h+1} = \sum_{i=0}^{h+1} a_i.\end{aligned}$$

□

Remark 5.15. Once again, the assumption that, in the lemma, the permutation f can be coded is fundamental. For instance, consider the series

$$1-1+\overbrace{\frac{1}{2}-\frac{1}{2}+\frac{1}{2}-\frac{1}{2}}^{4\text{ addenda}}+\cdots+\underbrace{\frac{1}{k}-\frac{1}{k}+\cdots+\frac{1}{k}-\frac{1}{k}}_{2k\text{ addenda}}+\cdots+\underbrace{\frac{1}{\infty_{-n-1}}-\frac{1}{\infty_{-n-1}}+\cdots+\frac{1}{\infty_{-n-1}}-\frac{1}{\infty_{-n-1}}}_{2\cdot\infty_{-n-1}\text{ addenda}}:$$

this is a series $\sum_{i=1}^{h} a_i$ of rational numbers $a_i \in \mathbb{Q}_n$ with $h \leq \infty_{-n}$ (if $1 \leq k \leq \infty_{-n-1}$ then $a_i = \frac{q}{\infty_{-n}}$, where q_i is the quotient of ∞_{-n} divided by k; moreover, $h = \infty_{-n-1}(\infty_{-n-1} + 1)$ as follows from $\sum_{i=1}^{y} 2 \cdot i = y \cdot (y + 1)$). It is easy to see that this series is definable, and its sum is 0.

On the other hand, the series can be reordered as

$$1+\frac{1}{2}+\frac{1}{2}-1+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}-\frac{1}{2}\dots+\underbrace{\frac{1}{k}+\frac{1}{k}+\dots+\frac{1}{k}}_{k\text{ times}}-\frac{1}{a_{k-2}}(\text{all finite }k)+\dots$$
$$+\quad\underbrace{\frac{1}{b}-\frac{1}{b}+\dots+\frac{1}{b}-\frac{1}{b}}_{2b\text{ addenda, every infinite }b\leq\infty_{-n-1}}\quad+\cdots,$$

where $-\frac{1}{a_k}$ is the k^{th} negative term in the original series. It is clear that the new series does not converge to a finite number: for instance this can be easily verified in a natural model of Γ_n, see Section 2.2.

5.5. **Representing real-valued functions: an example.** As observed before, we can represent a function f from $\mathbb{Q}_n$ into $\mathbb{Q}_n$ by means of its code f° on $[0, 2\cdot\infty_{-n}^2]$ defined for all $k \in [0, 2\cdot\infty_{-n}^2]$, by $f^\circ(k) = (f(^\circ k))^\circ$.

But if f does not map $\mathbb{Q}_n$ into itself, we may need larger numbers than those in $\mathbb{Q}_n$, to represent it.

We introduce an example to illustrate the situation.

Example 5.16. Suppose that $f(x) = e^x$. Then we may decide to consider only the values e^x when $x \in \mathbb{Q}_2$. There are many ways to approximate the values of e^x. For instance, we may take

$$e^x = \sum_{i=0}^{\infty_{-2}} \frac{x^i}{i!}.$$

Note that for some (finite or infinite) integer $b \in [-\infty_{-2}^2, \infty_{-2}^2]$, $x = \frac{b}{\infty_{-2}}$ and hence, for all $i \leq \infty_{-2}$,

$$\frac{x^i}{i!} = \frac{b^i}{\infty_{-2}^i \cdot i!}.$$

We prove that $\infty_{-2}^i \cdot i!$ divides ∞_{-1}. Indeed, $\infty_{-2}^i \cdot i!$ divides $\infty_{-2}^{\infty_{-2}} \cdot \infty_{-2}!$, and it is left to prove that $\infty_{-2}^{\infty_{-2}} \cdot \infty_{-2}!$ divides ∞_{-1}. Now

$$\infty_{-2}^{\infty_{-2}} \cdot \infty_{-2}! = \infty_{-2}^{\infty_{-2}} \cdot (P(\infty_{-2}))! \cdot \frac{\infty_{-2}}{2} \cdot 2;$$

moreover $\frac{\infty_{-2}}{2} \cdot 2$ divides $(P(\infty_{-2}))!$ (as $\frac{\infty_{-2}}{2} < P(\infty_2)$, and the quotient is clearly even) and hence $\infty_{-2}^{\infty_{-2}}$ divides $(P(\infty_{-2}))!^{\infty_{-2}}$; on the other hand $\infty_{-2}!$ divides $\infty_{-2}^{\infty_{-2}}$. It follows that $(\infty_{-2})^{\infty_{-2}} \cdot \infty_{-2}!$ divides $((\infty_{-2} \cdot P(\infty_{-2})!)^{\infty_{-2}} = \infty_{-1}$. Denoting the quotient of ∞_{-1} and $\infty_{-2}^i \cdot i!$ by q_i, we have

$$e^x = \frac{\sum_{i=0}^{\infty_{-2}} b^i \cdot q_i}{\infty_{-1}},$$

thus turning to $\mathbb{Q}_1$ for the representation.

This example shows that we have a general method to replace a significant part of the mathematics of the continuum by discrete mathematics.

References

[1] Y. D. Sergeyev, A new applied approach for executing computations with infinite and infinitesimal quantities, Informatica 19 (4) (2008) 567–596.

[2] Y. D. Sergeyev, Numerical point of view on calculus for functions assuming finite, infinite, and infinitesimal values over finite, infinite, and infinitesimal domains, Nonlinear Anal. 71 (12) (2009) e1688–e1707.

[3] Y. D. Sergeyev, Lagrange lecture: Methodology of numerical computations with infinities and infinitesimals, Rendiconti del Seminario Matematico dell'Università e del Politecnico di Torino 68 (2) (2010) 95–113.

[4] Y. D. Sergeyev, On accuracy of mathematical languages used to deal with the riemann zeta function and the dirichlet eta function, p-Adic Numbers, Ultram. Anal. Appl. 3 (2) (2011) 129–148.

[5] Y. D. Sergeyev, Numerical computations with infinite and infinitesimal numbers: Theory and applications, in: A. Sorokin, P. Pardalos (Eds.), Dynamics of Information Systems: Algorithmic Approaches, Vol. 51 of Springer Proceedings in Mathematics & Statistics, Springer Verlag, New York, 2013, pp. 1–66.

[6] Y. D. Sergeyev, Arithmetic of Infinity, Edizioni Orizzonti Meridionali, Cosenza, 2003.

[7] Y. D. Sergeyev, A. Garro, Observability of Turing machines: A refinement of the theory of computation, Informatica, Lith. Acad. Sci. 21 (3) (2010) 425–454.

[8] Y. D. Sergeyev, A. Garro, Single-tape and multi-tape Turing machines through the lens of the grossone methodology, The Journal of Supercomputing 65 (2) (2013) 645–663.

[9] E. Nelson, Predicative Arithmetic, Mathematical Notes, Princeton University Press, Princeton, 1986.

[10] A. Wilkie, J. Paris, On the scheme of induction for bounded arithmetic formulas, Ann. Pure Appl. Logic 35 (1987) 261–302.

[11] S. R. Buss, Bounded Arithmetic, Bibliopolis, Naples, 1986.

[12] H. Rogers, Jr., Theory of Recursive Functions and Effective Computability, McGraw-Hill, New York, 1967.

[13] E. Mendelson, Introduction to Mathematical Logic, 5th Edition, Chapman and Hall/CRC, Boca Raton, 2009.

[14] C. Smoryński, Logical Number Theory I, An Introduction, Universitext, Springer-Verlag, Berlin-Heidelberg-New York, 1991.

[15] M. Davis, Computability and Unsolvability, Dover, New York, 1982.

[16] R. I. Soare, Recursively Enumerable Sets and Degrees, Perspectives in Mathematical Logic, Omega Series, Springer Verlag, Heidelberg, 1987.

[17] G. Lolli, Metamathematical investigations on the theory of Grossone, to appear in Appl. Math. Comp.

[18] R. Margenstern, Using Grossone to count the number of elements of infinite sets and the connection with bijections, p-Adic Numbers, Ultram. Anal. Appl. 3 (3) (2011) 196–204.

[19] A. Zhigljavsky, Computing sums of conditionally convergent and divergent series using the concept of grossone, Appl. Math. Comp. 218 (16) (2012) 8064–8076.

Dipartimento di Ingegneria Informatica e Scienze Matematiche, Università Degli Studi di Siena, I-53100 Siena, Italy

E-mail address: franco.montagna@unisi.it

Dipartimento di Ingegneria Informatica e Scienze Matematiche, Università Degli Studi di Siena, I-53100 Siena, Italy

E-mail address: giulia.simi@unisi.it

Dipartimento di Ingegneria Informatica e Scienze Matematiche, Università Degli Studi di Siena, I-53100 Siena, Italy

E-mail address: andrea.sorbi@unisi.it