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Infinite Numerical Computing Applied to Hilbert's, Peano's, and Moore's Curves

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Abstract. The Peano and the Hilbert curves, denoted by P and H respectively, are historically the first and some of the best known space-filling curves. They have a fractal structure, many variants (as the well-known Moore curve M or a probably new "looped" version $\overline{H}$ of H), and a huge number of applications in the most diverse fields of mathematics and experimental sciences. In this paper, we employ a recently proposed computational system, allowing numerical calculations with infinite and infinitesimal numbers, to investigate the behavior of such curves and to highlight the differences with the classical treatment. In particular, we perform several types of computations and give many examples based not only on the curves H and P, but also on their d-dimensional versions H^d and P^d, respectively. Following our approach, it is easy to apply this new computational methodology to many other geometrical contexts, with interesting advantages such as summarizing in a single (infinite) number, representing the final result of a sequence of computations, much information both on the geometrical meaning of such a sequence and on the base geometrical structure itself.

Mathematics Subject Classification. Primary 03H15; Secondary 28A75.

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1. Introduction

The first space-filling curve was constructed by the Italian mathematician Giuseppe Peano in 1890 (see [28]), who gave a formal mathematical description both in the plane and in the ordinary space. But it was Hilbert, the following year (see [22]), who made the construction of such curves clear from a geometric point of view and paved the way to many generalizations and more than a century of intense studies and research.

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Today space-filling curves represent a very lively field of research both in pure mathematics and in computer science, physics, electronics, signal theory, engineering, optimization theory, parallel computing, and many others. In particular, we highlight the vast and important field of applications that the Hilbert curve finds in computer science because of its very good properties of locality: in fact, its discrete approximations, widely used in information technology, allow a correspondence between one- and multidimensional spaces preserving very well the proximity of the positions. For more details, applications and its numerous generalizations in dimension $d \geq 3$, the interested reader can see [1,2,4,19–21,33,34] (and the references therein) or Sect. 4.

The aim of our paper is to apply a new computational system, allowing precise and sharp *numerical calculations* with infinite and infinitesimal quantities, to Peano and Hilbert space-filling curves in any dimension $d \geq 2$. Following the examples and arguments presented here, it is not difficult to apply such a new framework to many other geometrical or mathematical contexts.

The mentioned computational system was introduced by Y.D. Sergeyev in the early 2000's: we refer the reader to [35,38,41,45] for detailed introductory surveys on the subject showing how to work numerically with infinite and infinitesimals numbers in a very easy and handy way, or to the book [36] as well, written in a popular manner. Such a new computational methodology has found recently many applications in a number of theoretical and computational research areas as optimization theory and numerical differentiation (see [15,18,27,40,53]), fractals (see [11,12,37,39,44,46]), hyperbolic geometry (see [26]), cellular automata and complex systems (see [16,17] and in the context of [5–7] under investigation), numerical series and Z-transform (see [14,52]), Hilbert problems, Turing machines and supertasks (see [30,32,42,47]), numerical solution of ordinary differential equations (see [27,40,43,48]), or even for speculative-didactic purposes (see [3,13,23,31]).

As regards the organization of the present paper, in Sect. 2 we first recall the construction of Peano and Hilbert curves in the plane and note how this construction essentially leaves no possibility of choice in 2 dimensions, while in higher dimensions there are many possible different configurations (see Sect. 4). Then we also recall an essential description of the extended sets $\widehat{\mathbb{N}}$, $\widehat{\mathbb{Z}}$, $\widehat{\mathbb{Q}}$ through grossone-based infinite numbers, and in such a context we give a new heuristic and handy definition of *elementary extended rational number* that could be useful in general for Sergeyev's theory (see Definition 2.1). Example 1 deals with such a new terminology, instead Examples 2–5 provide instances of different types of possible computations using the grossone-based framework on our curves. We have intentionally chosen to perform simple calculations to better highlight the differences obtained by using the new computational system with respect to traditional tools and classical mathematical analysis.

In Sect. 3, we give some examples of application of this new computational methodology for larger dimensional measures (in particular areas, but it is also possible to consider volumes and even b-dimensional volumes for every $0 \leq b \leq d$, as in [12]). In particular, we consider the *Moore curve* M, and apply the grossone system to calculations that start from the so-called *Moore polygons* $\mathcal{M}_n$. Since we have not been able to find references to useful formulas in the literature, we must first determine them. For this purpose we consider a "looped" version $\overline{H}$ of the Hilbert curve H: its approximations will give the desired formulas in a trivial way.

In Sect. 4, we consider the Peano and Hilbert curves, P^d and H^d respectively, in any dimension $d \geq 2$ and generalize some examples of Sect. 2 to highlight the role of the dimension when working geometrically with the grossone-based system and the dependence of the obtained results from d. In particular, the instances of series expansion in (4.11) and (4.12), together with (2.14) and (2.15) of Sect. 2, seem very interesting: we in fact remark that such series are sum of *true infinitesimal numbers*, on the contrary of what happens in classical analysis although it uses a similar terminology.[1]

Finally, Sect. 5 is devoted to conclusions.

We close the Introduction with a brief remark on the notations, but first we want to highlight two more important differences that arise from the new system and that we will encounter several times in all sections of the paper: using the grossone-based framework, the curves P^d and H^d do not fill any hypercube of dimension d, indeed, for each fixed d, these curves are no longer even unique, but in their place there exists a whole family of infinitely many curves "at infinity".

Regarding notations, as usual we denote a sequence with $\{a_n\}_n$, $\{a_n\}$, or sometimes just a_n. We use the symbol $\mathbb{N}$ for the set of natural numbers ($0 \notin \mathbb{N}$), and $\mathbb{N}_0$ for $\mathbb{N} \cup \{0\}$.

2. The Hilbert and Peano Curves in the Plane

2.1. Traditional Constructions

There are several ways to construct the Hilbert and the Peano curves in the plane. If we denote by I the unit interval $[0, 1] \subset \mathbb{R}$, one of the simplest ways for the Hilbert curve, is to consider a family of continuous functions $\{H_n : I \to I^2 = [0,1] \times [0,1] \subset \mathbb{R}^2 \,|\, n \in \mathbb{N}_0\}$, where the image of H_n is a broken line contained in the unit square I^2. It is necessary, for the computational purposes of the next sections, to briefly recall and fix its construction because there are different configurations or variations of both the Hilbert curve and the Peano one.

First of all, by convenience, we set $h := 1/2$ and let θ be the anticlockwise rotation of a figure by an angle of $\pi/2$ around its center. We start by setting $H_0 := (1/2, 1/2)$, i.e., the central point of I^2, and we construct the broken line H_1 as in Fig. 1a: note that it connects the centers of the four smaller squares in which I^2 is divided. To get H_2, we use $\theta^{-1}(H_1)$, two copies of H_1 and one of $\theta(H_1)$, but all them scaled by $h = 1/2$: we display

[1]The basic proposition of elementary calculus often called *necessary Cauchy condition for convergence* states that "if $\sum_n a_n$ converges, then $\{a_n\}_n$ is *infinitesimal*". But the word "infinitesimal" does not refer, obviously, to the elements a_n which are, in this case, ordinary real or complex numbers.

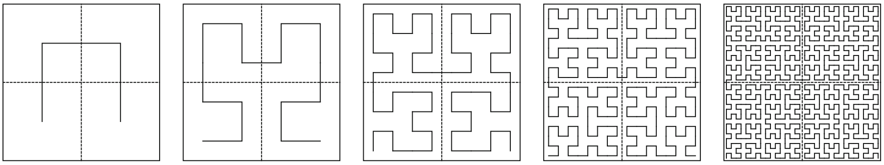
(a) H_1 **(b)** H_2 **(c)** H_3 **(d)** H_4 **(e)** H_5

Figure 1. The first five steps in the construction of the Hilbert curve starting from the configuration H_1

and we connect them as in Fig. 1b, by inserting 3 line segments of length h^2. Then we iterate the process: to get H_{n+1} we use $\theta^{-1}(H_n)$, two copies of H_n and one of $\theta(H_n)$, all scaled by 1/2 again. Hence, we arrange them in the obvious way, clockwise as before, and we join such copies by adding 3 segments of length h^{n+1}. The *Hilbert curve* $H : I \to I^2$ can be defined as the limit $H := \lim_{n\to\infty} H_n$ in the usual sense of fractal geometry.

To construct our version of the Peano curve, we pose $k := 1/3$ to shorten writings and, above all, to emphasize the role of the constants h and k for H and P, respectively. Let P_0 be the central point $(1/2, 1/2)$ of I^2, then we divide I^2 into 9 smaller squares of side k and we connect the centers of the squares as in Fig. 2a to obtain the broken line P_1. Now let's denote by ρ the reflection around the horizontal axis of a figure; using 5 copies of P_1 and 4 of $\rho(P_1)$, all scaled by 1/3, we obtain the broken line P_2 as in Fig. 2b. As before, we iterate the process: to obtain P_{n+1}, we use 5 copies of P_n and 4 of $\rho(P_n)$ always scaled by 1/3, and we dispose them alternatively into the 9 squares of side 1/3 composing I^2, following the order "first the left 3 squares from below to above, then the 3 central squares from above to below, and finally the right 3 squares from below to above." Then we connect them by adding 8 line segments of length k^{n+1}. Finally, the *Peano curve* $P : I \to I^2$ is the limit $P := \lim_{n\to\infty} P_n$. It is well known that both curves H and P are continuous, nowhere differentiable and surjective on the unit square.

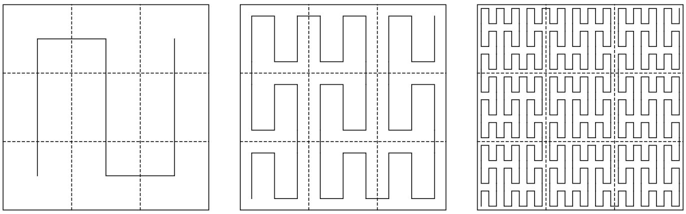
(a) P_1 **(b)** P_2 **(c)** P_3

Figure 2. The first three steps in the construction of the Peano curve starting from P_1. Note as, obviously, the sequence P_n is much "faster" than H_n to fill the unit square

The necessity of giving precise definitions of H and P emerges clearly even from the following simple observation: if we use, for example, the reflection ρ' around the vertical axis in the place of ρ, we obtain a twin curve P' different from P.

If we denote by $l(C)$ the length of a curve C, from the construction of H_n we get the easy recursive formula

$$l(H_{n+1}) \;=\; 4 \cdot \frac{l(H_n)}{2} + 3 \cdot h^{n+1} = 4 \cdot \frac{l(H_n)}{2} + 3 \cdot \left(\frac{1}{2}\right)^{n+1} \tag{2.1}$$

holding for all $n \in \mathbb{N}_0$, and from which we obtain, by induction,

$$l(H_n) = 2^n - \left(\frac{1}{2}\right)^n, \qquad n \geq 0. \tag{2.2}$$

Another simpler way to obtain (2.2) is to note that H_n is constituted by $4^n - 1$ segments of length $h^n = (1/2)^n$ (i.e., one for each small square in which I^2 is divided at the n-th step, minus 1).

For the Peano curve P we obtain similar formulas and, in particular, (2.2) becomes

$$l(P_n) = 3^n - \left(\frac{1}{3}\right)^n, \qquad n \geq 0. \tag{2.3}$$

Taking limits in (2.2) and (2.3) we immediately get

$$l(H) = +\infty \qquad \text{and} \qquad l(P) = +\infty, \tag{2.4}$$

i.e., the Hilbert and Peano curves have infinite length.

2.2. The Grossone-Based Framework

From now onwards, we assume that the reader is familiar with Sergeyev's grossone-based system and is able to perform computations with it (see [35, 36, 38, 41, 45] for introductive or general references). We just recall that ① is the number of elements of $\mathbb{N}$ and every sequence (finite or infinite) has length at most $\leq$ ①. Moreover, the *extended set of natural numbers*, denoted by $\widehat{\mathbb{N}}$, can be written as

$$\begin{aligned}\widehat{\mathbb{N}} = \Big\{ & \underbrace{1, 2, \ldots, ① - 1, ①}_{\mathbb{N}}, ① + 1, \ldots, 2①, \ldots, 3①, \ldots, \\ & ①^2, ①^2 + 1, \ldots, ①^2 + ①, \ldots, 2①^2, \ldots, ①^3, \ldots, \\ & 2^{①}, \ldots, 2^{①} + ①, \ldots, 2^{①} + ①^2, \ldots, 3^{①}, \ldots, ①^{①}, \ldots \Big\},\end{aligned}$$

the *extended set of integers* $\widehat{\mathbb{Z}} = \widehat{\mathbb{N}} \cup \{0\} \cup \{-N : N \in \widehat{\mathbb{N}}\}$, and we also set $\widehat{\mathbb{N}}_0 = \widehat{\mathbb{N}} \cup \{0\}$. The *extended set of rational numbers*, denoted by $\widehat{\mathbb{Q}}$, is the quotient field of $\widehat{\mathbb{Z}}$, and this set is almost all we need for the purposes of the present paper.

For convenience, the next definition introduces a new nomenclature in a heuristic and not too formal way.

Definition 2.1. An extended rational number $\alpha \in \widehat{\mathbb{Q}}$ is called *elementary*, if it is expressible without using the sum sign/operation. Otherwise, it is called *non-elementary*.

Example 1. For instance,

$$\frac{6}{11}, \quad ①^4, \quad -\frac{①}{3}, \quad \left(\frac{4}{7}\right)^{-4①}, \quad \left(\frac{3}{2①}\right)^{4①}, \quad \frac{(5/4)^{①/2}}{3①^8}$$

are all, of course, elementary (extended) rational numbers, whereas

$$\frac{2+①}{3}, \qquad ①^4-6, \qquad \frac{①+2}{3}, \qquad \frac{3^{①}}{2①-3}$$

are not elementary: in fact, each of them is a sum of (at least) two elementary numbers. Note, moreover, that a number as $(2/3)^{①^3-2①-5}$ is elementary because it can be written in the form $(2/3)^{①^3} \cdot (2/3)^{-2①} \cdot (2/3)^{-5}$.

First of all, we point out that using the grossone system, we have no longer a unique limit curve H and P, but many different curves depending on the initial and final value of the previously seen constructions $\{H_n\}_n$ and $\{P_n\}_n$, respectively. Specific references about this phenomenon in other contexts are [11,12,39,44,46]. Using, if necessary, a chain of many sequences, one starting at the end point of another, we obtain curves H_N and P_N where N can be an arbitrarily large infinite positive integer written in the new system.

Example 2. If we consider ① steps in the construction of H, i.e., in the sequence $\{H_n\}_n$, we obtain a curve which we can denote by $H_{①}$. Its length is $2^{①}-2^{-①}$ which is an infinite number much larger than ①. If we start from $H_{①}$, and we iterate the process, say, for other 3 steps, we obtain a different curve which we can denote by $H_{①+3}$, according to the notations we are using, and whose length is $2^{①+3}-2^{-①-3}$.
It is obvious, but important to remark, that, if we start the construction $\{H_n\}_n$ from H_3, i.e., the arrangement of Fig. 1c, after a chain of ① steps we obtain the curve $H_{①+3}$ itself.

In the same way, we find that the curve $P_{①}$ has length $3^{①}-3^{-①}$ and, in general, for any (finite or) infinite non-negative integer N, P_N has length 3^N-3^{-N}.

Example 3. If we look for a way to evaluate the difference between the lengths of the curves H and P, (2.4) is not useful because it lost all the information about them and their constructive process. All we can get from (2.4) is, in fact, the following indeterminate expression

$$l(H)-l(P) \;=\; +\infty-(+\infty). \tag{2.5}$$

Instead, if we use the grossone system and consider, for instance, $H_{①}$ and $P_{①}$, the difference between their lengths is

$$l\left(H_{①}\right)-l\left(P_{①}\right)=2^{①}-\left(\frac{1}{2}\right)^{①}-3^{①}+\left(\frac{1}{3}\right)^{①}$$

$$= -\left(3^{①} - 2^{①}\right)\left(1 + \frac{1}{6^{①}}\right). \tag{2.6}$$

Similarly, from (2.4) we can just achieve the indeterminate form

$$\frac{l(P)}{l(H)} = \frac{+\infty}{+\infty}$$

for the ratio of $l(P)$ and $l(H)$. Using instead the new system and considering $N = ①$ steps, we have

$$\frac{l\left(P_{①}\right)}{l\left(H_{①}\right)} = \frac{3^{①} - 3^{-①}}{2^{①} - 2^{-①}} \tag{2.7}$$

$$= \frac{6^{①}}{4^{①} - 1} - \frac{2^{①}}{3^{①}\left(4^{①} - 1\right)}. \tag{2.8}$$

Note that in (2.8) the first summand is an infinite number, the second an infinitesimal one. It is also clear that $l\left(P_{①}\right)/l\left(H_{①}\right)$ is equal to $(3/2)^{①}$ up to infinitesimals.[2]

Example 4. In the new framework, we can also compare two different infinite steps of a process; for instance, if we consider the curves $H_{①}$ and $H_{①+3}$ of Example 2, we have

$$l\left(H_{①+3}\right) - l\left(H_{①}\right) = 2^{①+3} - 2^{-①-3} - 2^{①} + 2^{-①} \tag{2.9}$$

$$= 7 \cdot 2^{①} + \frac{7}{8} \cdot 2^{-①} \tag{2.10}$$

and

$$\frac{l\left(H_{①+3}\right)}{l\left(H_{①}\right)} = \frac{2^{①+3} - 2^{-①-3}}{2^{①} - 2^{-①}} \tag{2.11}$$

$$= \frac{8 - \frac{1}{8} \cdot 4^{-①}}{1 - 4^{-①}}, \tag{2.12}$$

which represent the precise value of the difference and of the ratio of their lengths, respectively. Note that, as we can expect, the ratio $l\left(H_{①+3}\right)/l\left(H_{①}\right)$ is "infinitely close" to 8.

Example 5. We would like to express (2.8) and (2.12) in a better way, as sums of elementary numbers like (2.6) or (2.10). In fact, some information are rather hidden in (2.8) and (2.12); for example, consider the following questions:

(i) If we substitute $l\left(P_{①}\right)/l\left(H_{①}\right)$ with $(3/2)^{①}$, what error are we making? Is it greater or less than $(1/2)^{①}$?

(ii) As in (i), what error do we make by replacing $l\left(H_{①+3}\right)/l\left(H_{①}\right)$ with 8? Is it greater or less than $(1/4)^{①}$?

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[2] Note that $\dfrac{6^{①}}{4^{①} - 1} = \left(\dfrac{3}{2}\right)^{①} + \dfrac{(3/2)^{①}}{4^{①} - 1}.$

It is very interesting to develop (2.8) and (2.12) as power series, or better as infinite sum of grossone-based powers. In the first case, we obtain

$$\frac{l\left(P_{①}\right)}{l\left(H_{①}\right)} = \frac{3^{①} - 3^{-①}}{2^{①} - 2^{-①}}$$

$$= \left(\frac{3}{2}\right)^{①} + \sum_{n\geq 0} \left(\frac{3}{8\cdot 4^{n}}\right)^{①} - \left(\frac{1}{6\cdot 4^{n}}\right)^{①} \tag{2.13}$$

$$= \left(\frac{3}{2}\right)^{①} + \left[\left(\frac{3}{8}\right)^{①} - \left(\frac{1}{6}\right)^{①}\right] \cdot \sum_{n\geq 0} \left(\frac{1}{4^{①}}\right)^{n}. \tag{2.14}$$

Note that (2.14) is an immediate consequence of (2.13), instead, to prove the truth of the expression (2.13) itself, one can multiply it by $l\left(H_{①}\right) = 2^{①} - 2^{-①}$, obtaining, in this manner, $l\left(P_{①}\right) = 3^{①} - 3^{-①}$. Another method to show (2.13) is by using the division algorithm, well known from basic algebra of polynomials and formal power series. In such a way, we can directly compute the quotient $\left(3^{①} - 3^{-①}\right)/\left(2^{①} - 2^{-①}\right)$ in the form of an infinite sum of powers (see [11] for other details and examples on this method).

With similar computations as in (2.13) and (2.14), we obtain

$$\frac{l\left(H_{①+3}\right)}{l\left(H_{①}\right)} = \frac{2^{①+3} - 2^{-①-3}}{2^{①} - 2^{-①}}$$

$$= 8 + \frac{63}{8} \sum_{n\geq 1} \left(\frac{1}{4^{①}}\right)^{n}. \tag{2.15}$$

Now it is very simple to answer to (i) and (ii): from (2.13) we have immediately that the error in the approximation of $l\left(P_{①}\right)/l\left(H_{①}\right)$ considered above is about $(3/8)^{①}$ $\left(\text{hence, less than} (1/2)^{①}\right)$. For the second question, we get from (2.15) that the error in the approximation of $l\left(H_{①+3}\right)/l\left(H_{①}\right)$ is about $7.875 \cdot (1/4)^{①}$ $\left(\text{hence, greater than} (1/4)^{①}\right)$.

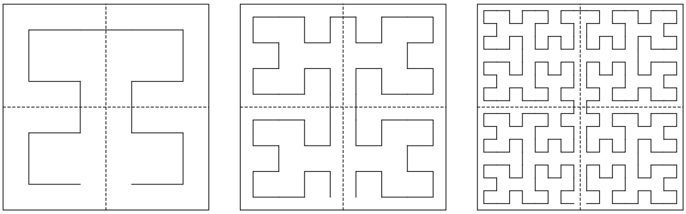
3. The Moore Curve and the Area of Moore Polygons

3.1. Traditional Framework

The Moore curve is an interesting variant of the Hilbert curve emanating from the point $(1/2, 0)$ instead of $(0, 0)$ and $(1, 0)$. Keeping well in mind the notations used at the beginning of Sect. 2, we can easily define a sequence of curves $\left\{M_n : I \to I^2 \subset \mathbb{R}^2 \,|\, n \in \mathbb{N}_0\right\}$ where $M_0 := H_0 = (1/2, 1/2)$, $M_1 := H_1$ and, for all $n \geq 2$, M_n consists of two copies of $\theta(H_{n-1})$ and two of $\theta^{-1}(H_{n-1})$, displaced in clockwise order and connected by adding three line segments of length $h^n = (1/2)^n$ as Fig. 3a–c show when $n = 2,\ 3$ and 4, respectively. Therefore, the Moore curve M is the limit $\lim_{m\to\infty} M_n$.

If $P = (x_1, \ldots, x_n)$ and $Q = (y_1, \ldots, y_n)$ are any two points of $\mathbb{R}^n$, $n \geq 2$, we recall that $[P, Q]$ usually denotes the parameterized line segment from P to Q, that is

$$\begin{aligned}[P, Q] &:= \left\{P + t(Q - P) \,|\, t \in [0, 1]\right\} \\ &= \left\{\left(x_1 + t(y_1 - x_1), \ldots, x_n + t(y_n - x_n)\right) \,|\, t \in [0, 1]\right\}.\end{aligned}$$



(a) M_2 **(b)** M_3 **(c)** M_4

Figure 3. Three consecutive steps in the construction of the Moore curve

Moreover, if $\gamma_1, \gamma_2 : [0,1] \to \mathbb{R}^n$ are two paths with $\gamma_1(1) = \gamma_2(0)$, we denote by $\gamma_1 * \gamma_2 : [0,1] \to \mathbb{R}^n$ the usual *path product* defined by

$$(\gamma_1 * \gamma_2)(t) := \begin{cases} \gamma_1(2t) & \text{if } \ 0 \le t \le 1/2, \\ \gamma_2(2t-1) & \text{if } \ 1/2 \le t \le 1 \end{cases}$$

(see, e.g., [50] or any book of algebraic topology). For the accuracy, we also remind readers that "$*$" is not associative and, therefore, when we will write a product of three or more paths without using brackets, it will mean that the products must be executed in the order in which they occur.

As we said at the beginning of the section, M is a closed curve but the M_n are not. Adding to the curve M_n, $n \ge 1$, the further (parameterized) segment

$$\left[\left(h - h^{n+1}, h^{n+1}\right), \left(h + h^{n+1}, h^{n+1}\right)\right] = \left[\left(\tfrac{2^n-1}{2^{n+1}}, \tfrac{1}{2^{n+1}}\right), \left(\tfrac{2^n+1}{2^{n+1}}, \tfrac{1}{2^{n+1}}\right)\right]$$

of length $h^n = 1/2^n$, we obtain a *closed polygonal path* denoted by $\overline{M}_n$. We moreover set $\mathcal{M}_n := \overline{M}_n(I) \cup \operatorname{Int}\left(\overline{M}_n\right)$, where $\overline{M}_n(I)$ is the image and $\operatorname{Int}\left(\overline{M}_n\right)$ the *interior* of $\overline{M}_n$, respectively. Hence we call the $\mathcal{M}_n$ *Moore polygons* (see also [51, Chap. VIII]) and note that the limit $\lim_{n\to\infty} \overline{M}_n$ is M as well; in Fig. 4a, b, $\mathcal{M}_3$ and $\mathcal{M}_4$ are represented in light purple, respectively.

As regards the length of M_n and $\overline{M}_n$, we trivially have

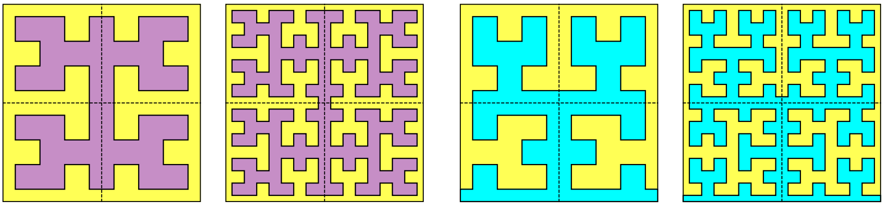
$$l\left(M_n\right) = l\left(H_n\right) = 2^n - \frac{1}{2^n} \qquad \text{and} \qquad l\left(\overline{M}_n\right) = l\left(H_n\right) + h^n = 2^n, \quad (3.1)$$

instead the area of the *simple polygon* $\mathcal{M}_n$ seems more enticing and unnoticed in literature. The simplest way to compute the area $A\left(\mathcal{M}_n\right)$ of $\mathcal{M}_n$ arises very easily after defining a new family of simple polygons $\left\{\mathcal{H}_n \subseteq I^2 \mid n \in \mathbb{N}_0\right\}$, where the boundary $\overline{H}_n$ of $\mathcal{H}_n$, as parametric curve, is the product of the following six paths

$$\begin{aligned} \overline{H}_n := & \left[(0,0), \left(0, h^{n+1}\right)\right] * \left[\left(0, h^{n+1}\right), \left(h^{n+1}, h^{n+1}\right)\right] \\ & * H_n * \left[\left(1 - h^{n+1}, h^{n+1}\right), \left(1, h^{n+1}\right)\right] \\ & * \left[\left(1, h^{n+1}\right), (1,0)\right] * \left[(1,0), (0,0)\right], \end{aligned}$$

for every integer $n \ge 0$; note that its length is hence (see (2.2))

$$l\left(\overline{H}_n\right) = 2 \cdot \left(\frac{1}{2}\right)^{n+1} + l\left(H_n\right) + 2 \cdot \left(\frac{1}{2}\right)^{n+1} + 1 = 2^n + 1 + \frac{1}{2^n}. \quad (3.2)$$



(a) $\mathcal{M}_3$ **(b)** $\mathcal{M}_4$ **(c)** $\mathcal{H}_3$ **(d)** $\mathcal{H}_4$

Figure 4. In **a**, **b**, the Moore polygons $\mathcal{M}_3$ and $\mathcal{M}_4$, which correspond to the curves M_3 and M_4 of Fig. 3b, c, are pictured, respectively. **c** shows in cyan the polygon $\mathcal{H}_3$: note that the cyan area in the upper right quadrant, if rotated by an angle of $-\pi/2$, can be perfectly overlapped on the yellow area in the lower left quadrant, and similarly for the upper left and the lower right quadrants. Hence $A(\mathcal{H}_3) = 1/2$ and, in the same way, we can conclude that the area of the polygon $\mathcal{H}_4$ shown in **d** is 1/2 too

$\mathcal{H}_3$ and $\mathcal{H}_4$ are represented in cyan in Fig. 4c, d, respectively, and looking also at such particular cases, it is clear that the area $A(\mathcal{H}_n)$ of $\mathcal{H}_n$ does not depend on n and is equal to the recurring constant h, i.e., in symbols

$$A(\mathcal{H}_n) = \frac{1}{2} \qquad \text{for all} \quad n \in \mathbb{N}_0. \tag{3.3}$$

We are now in a position to determine the area of the n-th Moore polygon $\mathcal{M}_n$. From the definition of M_n it is easy to note that if we add to $\mathcal{M}_n$, $n \geq 1$, the following two small rectangles

$$\left[h - h^{n+1}, h + h^{n+1}\right] \times \left[0, h^{n+1}\right] \quad \text{and} \quad \left[h - h^{n+1}, h + h^{n+1}\right] \times \left[1 - h^{n+1}, 1\right],$$

then we obtain a figure that consists exactly of four copies of $\mathcal{H}_{n-1}$ scaled by 1/2 (see for example Fig. 4b, c). Hence we get

$$A(\mathcal{M}_n) = 4 \cdot \frac{A(\mathcal{H}_{n-1})}{4} - 4 \cdot \left(h^{n+1}\right)^2 = \frac{1}{2} - \frac{1}{4^n} \tag{3.4}$$

for all $n \geq 1$.

3.2. New Point of View: Advantages Provided by the Use of the Grossone Framework

For any infinite number N belonging to the grossone system, we can construct, in analogy with Example 2 , the curves M_N, $\overline{M}_N$, $\overline{H}_N$, and the polygons $\mathcal{M}_N$ and $\mathcal{H}_N$. A first important discrepancy with classical analysis is that none of the curves $M_{①}$, $\overline{M}_{①}$, $\overline{H}_{①}$, or M_N, $\overline{M}_N$, $\overline{H}_N$ for any infinite positive integer N, fills the square $[0,1] \times [0,1]$ if we consider the interval

$[0, 1]$ in the new system too.[3] And the same is the case for the polygons $\mathcal{M}_N$ and $\mathcal{H}_N$ for all $N \in \widehat{\mathbb{N}}$.

Considering now the length of M, $\overline{M}$ and $\overline{H}$ in classical analysis, by taking limits in (3.1) and (3.1) we obtain

$$l(M) \;=\; l\left(\overline{M}\right) \;=\; l\left(\overline{H}\right) \;=\; +\infty, \tag{3.5}$$

which would means that M, $\overline{M}$ and $\overline{H}$ have "the same infinite length." Note in fact that we usually do not distinguish a writing like (3.5) from the following

$$l(M) = +\infty, \qquad l\left(\overline{M}\right) = +\infty, \qquad l\left(\overline{H}\right) = +\infty, \tag{3.6}$$

giving both them the same meaning. Such an example with curves can be taken as a prototype of a large number of more or less similar circumstances occurring in contemporary mathematics and in all fields where it finds application. Whenever there is a need to distinguish and to make computations with infinite quantities, Sergeyev's system could give a useful support.

Let us now consider a sequence of, for instance, ① steps in the construction of M, $\overline{M}$ and $\overline{H}$; from (3.1) and (3.2) we obtain

$$l\left(M_{①}\right) = 2^{①} - \frac{1}{2^{①}}, \quad l\left(\overline{M}_{①}\right) = 2^{①}, \quad l\left(\overline{H}_{①}\right) = 2^{①} + 1 + \frac{1}{2^{①}}, \tag{3.7}$$

which can be generalized in the obvious way to any N belonging to the extended set $\widehat{\mathbb{N}}$. From (3.7), it is clear that using the new framework, these measures are distinct, even if only slightly, in dependence on the constructive algorithm and the geometry of the curve.

The use of the new computational methodology is different from the mechanical replacement of the symbol ∞ with ① or with some writing containing it, as the following example shows.

Example 6. Consider the sequences of images

$$\left\{\overline{M}_n(I)\right\}_n \qquad \text{and} \qquad \left\{\overline{H}_n(I)\right\}_n \tag{3.8}$$

as subsets of I^2. As n grows (among finite naturals), the sequences in (3.8) produce an ever more complex set with fractal characteristics, but when we take the limit $n \to +\infty$ we obtain a simple regular object, the whole square I^2, which covers a unit area as expressed in symbols below:

$$A\left(\lim_{n\to+\infty} \overline{M}_n(I)\right) \;=\; A\left(\lim_{n\to+\infty} \overline{H}_n(I)\right) \;=\; A\left(I^2\right) \;=\; 1. \tag{3.9}$$

On the contrary, instead, for $n =$ ① the sequences (3.8) originate a pair of extreme fractal objects with some surprising features: for instance, they are curves with sides of infinitesimal length but at the same time infinitely long and with a "piecewise regularity" in open contrast with $\overline{M}$ and $\overline{H}$, etc. Moreover, it provides a clear example of how the simple replacement of the

[3] We give here some explanations. First of all, for any $\varepsilon > 0$ and $P \in \mathbb{R}^2$ denote by $D_\varepsilon(P)$ the open disc of radius ε centered at P. Consider, for instance, the curve M_N, with N any number of $\widehat{\mathbb{N}}$, and a point $P \in M_N$. Then note that $M_N \cap D_\varepsilon(P) \subseteq L_h \cup L_v$, where L_h and L_v are two suitable lines of $\mathbb{R}^2$ (L_h horizontal and L_v vertical, and one of them at least contains P).

limits in (3.9) by the evaluation $n = ①$ yields wrong results, because we actually have

$$A\left(\overline{M}_{①}(I)\right) \;=\; A\left(\overline{H}_{①}(I)\right) \;=\; 0, \tag{3.10}$$

and more in general, obviously,

$$A\left(\overline{M}_N(I)\right) \;=\; A\left(\overline{H}_N(I)\right) \;=\; 0 \qquad \text{for all} \;\; N \in \widehat{\mathbb{N}}. \tag{3.11}$$

A further singular thing is that both $\overline{M}_N(I)$ and $\overline{H}_N(I)$ are boundaries of subsets of I^2, $\mathcal{M}_N$ and $\mathcal{H}_N$, respectively, having approximately the same area equal to 1/2 (hence different from the result in (3.9) too). More precisely, we get (see also (3.4))

$$A\left(\mathcal{M}_N\right) = \frac{1}{2} - \frac{1}{4^N} \qquad \text{and} \qquad A\left(\mathcal{H}_N\right) = \frac{1}{2} \qquad \text{for all } N \in \widehat{\mathbb{N}}. \tag{3.12}$$

For instance, the last equation implies that the area of the polygon obtained at any step of the construction of $\mathcal{H}_{①}$ is always constant and precisely equal to 1/2.

4. The Hilbert and the Peano Curves in Higher Dimension

Unlike the two-dimensional setting in which the Hilbert curve H is essentially unique (i.e., unique except for rotations and reflections, or in other words, except for an element of the dihedral group D_8 of the symmetries of the square), there are many different non-equivalent constructions in higher dimensions, for example, 1536 or as many as 10 694 807 in dimension 3, depending on the precise definition and on the desired properties characterizing a three-dimensional "Hilbert curve" (see [1], [4, Chap. 8], [19,20] and the references therein). For $d = 4$ instead, things become much more complicated and it is an open problem in many cases.

In this section, we want to give some examples of computation on space-filling curves in dimension $d > 2$ using the grossone system; in particular we will consider the length of such curves in $\mathbb{R}^d$. Since every commonly used construction of the Hilbert or the Peano curve filling the d-dimensional unitary cube $I^d \subset \mathbb{R}^d$ has the same length at each step n, we can fix any particular sequence of curves $\left\{H_n^d : I \to I^d \,|\, n \in \mathbb{N}_0\right\}$ and $\left\{P_n^d : I \to I^d \,|\, n \in \mathbb{N}_0\right\}$ for the Hilbert and the Peano d-dimensional curve, respectively (there are many references in literature for such and similar constructions, generating algorithms and discussions of the resulting properties, as for instance [1,2,8–10,24,25,29,33,34,49]). Hence, as in the 2-dimensional case, we set $H^d = \lim_{n\to\infty} H_n^d$ and $P^d = \lim_{n\to\infty} P_n^d$: we will therefore understand these curves when, in the following, we will talk about the *d-dimensional Hilbert curve* and the *d-dimensional Peano curve*. Obviously, choosing the previous families of curves, we require to recover, for $d = 2$, the sequences of Sect. 2, i.e., $H_n^2 = H_n$ and $P_n^2 = P_n$ for all $n \in \mathbb{N}_0$, so that it yields $H^2 = H$ and $P^2 = P$.

As regards the length of H_n^d note that, since I^d is divided into 2^{nd} small d-cubes of side $h^n = (1/2)^n$, then

$$l\big(H_n^d\big) \;=\; \frac{2^{nd}-1}{2^n}, \tag{4.1}$$

for all integers $d \geq 2$ and $n \geq 0$. Similarly, since P_n^d connects the centers of 3^{nd} small d-cubes of side $k^n = (1/3)^n$, in which is partitioned I^d, then

$$l\big(P_n^d\big) \;=\; \frac{3^{nd}-1}{3^n}, \tag{4.2}$$

for all integers $d \geq 2$ and $n \geq 0$. Note that, for $d = 2$ we recover (2.2) and (2.3), and taking limits in (4.1) and (2.2) we get the analog of (2.4), i.e.,

$$l\big(H^d\big) = +\infty, \qquad l\big(P^d\big) = +\infty \qquad \text{for all } d \geq 2. \tag{4.3}$$

At this point, as in Sect. 2, it is easy to perform computations using Sergeyev's system.

Example 7. Considering the lengths of $H_{①}^d$ and $P_{①}^d$ after ① steps, we get

$$l\big(H_{①}^d\big) = 2^{(d-1)①} - 2^{-①} \qquad \text{and} \qquad l\big(P_{①}^d\big) = 3^{(d-1)①} - 3^{-①} \tag{4.4}$$

for all finite integer $d \geq 2$. Moreover, reasoning as usual in case of a number of steps greater than ①, we can extend (4.3) in the obvious way, obtaining

$$l\big(H_N^d\big) = 2^{(d-1)N} - 2^{-N} \qquad \text{and} \qquad l\big(P_N^d\big) = 3^{(d-1)N} - 3^{-N} \tag{4.5}$$

for all finite $d \geq 2$ and $N \in \widehat{\mathbb{N}}$.

In the next three examples, we will show some simple cases of application of the previous formulas, following roughly what we did in Sect. 2 for the two-dimensional case.

Example 8. Notice how starting from (4.3) no numerical calculation is possible because indeterminate forms would be obtained as in (2.5). Instead, using (4.4) or (4.5) we can easily compare various lengths such as, for example,

$$l\big(P_{①}^d\big) - l\big(H_{①}^d\big) \;=\; 3^{(d-1)①} - 2^{(d-1)①} + \frac{1}{2^{①}} - \frac{1}{3^{①}} \qquad \forall d \geq 2 \tag{4.6}$$

(note that (4.6) generalizes trivially (2.6), if we reverse the sign). On the contrary of expressions like (2.5), from (4.6), it is immediately clear how large such a difference is. For instance, if one wanted to approximate $l\big(P_{①}^d\big)$ with some $l\big(H_N^t\big)$ for a certain dimension $t \geq 2$ and some infinite number $N \in \widehat{\mathbb{N}}$, he would easily get, recalling (4.5), that $l\big(H_N^t\big)$ gives good approximations of $l\big(P_{①}^d\big)$ when $\frac{(t-1)N}{(d-1)①}$ is very close to $\log_2 3$.[4]

[4]The interested reader can easily and in different ways give more formal and precise meanings to the term "approximation" used here; for instance, he could see [12, Definition 2.1] or simply start from the observation

$$l\big(H_N^t\big) = l\big(P_M^d\big) \quad \text{(up to infinitesimals)} \qquad \Leftrightarrow \qquad \frac{(t-1)N}{(d-1)M} = \log_2 3.$$

A second example could concern the ratio of lengths in (4.4) as we did in (2.7)–(2.8) for $d = 2$:

$$\frac{l\left(P^d_{①}\right)}{l\left(H^d_{①}\right)} = \frac{3^{(d-1)①} - 3^{-①}}{2^{(d-1)①} - 2^{-①}}$$

$$= \left(\frac{2}{3}\right)^{①} \cdot \frac{3^{d①} - 1}{2^{d①} - 1} \tag{4.7}$$

$$= \frac{\left(2 \cdot 3^{d-1}\right)^{①}}{2^{d①} - 1} - \frac{2^{①}}{3^{①}\left(2^{d①} - 1\right)}. \tag{4.8}$$

From (4.8) it follows immediately that $l\left(P^d_{①}\right)/l\left(H^d_{①}\right)$ is equal to $(3/2)^{(d-1)①}$ up to infinitesimals (recall Example 3, and in particular fn. 2).

Example 9. Using (4.5) we can easily compare the length of the curve obtained from two different infinite numbers of steps. For instance,

$$l\left(H^d_{①+3}\right) - l\left(H^d_{①}\right) = 2^{(d-1)(①+3)} - 2^{-(①+3)} - 2^{(d-1)①} + 2^{-①}$$

$$= \left(2^{3d-3} - 1\right) \cdot 2^{(d-1)①} + \frac{7}{8} \cdot 2^{-①} \tag{4.9}$$

and

$$\frac{l\left(H^d_{①+3}\right)}{l\left(H^d_{①}\right)} = \frac{2^{(d-1)(①+3)} - 2^{-(①+3)}}{2^{(d-1)①} + 2^{-①}}$$

$$= \frac{1}{8} \cdot \frac{8^d \cdot 2^{d①} - 1}{2^{d①} - 1} \tag{4.10}$$

are computations analogous to (2.9)–(2.12) in dimension d.

Example 10. We conclude by giving, as example, the generalizations of (2.14) and (2.15) for any dimension d:

$$\frac{l\left(P^d_{①}\right)}{l\left(H^d_{①}\right)} = \frac{3^{(d-1)①} - 3^{-①}}{2^{(d-1)①} - 2^{-①}}$$

$$= \left(\frac{3}{2}\right)^{(d-1)①} + \left(\frac{2}{3}\right)^{①}\left[\left(\frac{3}{4}\right)^{d①} - \left(\frac{1}{2}\right)^{d①}\right] \cdot \sum_{n \geq 0}\left(\frac{1}{2^{d①}}\right)^{n} \tag{4.11}$$

and

$$\frac{l\left(H^d_{①+3}\right)}{l\left(H^d_{①}\right)} = \frac{2^{(d-1)(①+3)} - 2^{-(①+3)}}{2^{(d-1)①} + 2^{-①}}$$

$$= \frac{1}{8} + \frac{8^d - 1}{8} \cdot \sum_{n \geq 0}\left(\frac{1}{2^{2d①}}\right)^{n}. \tag{4.12}$$

Choosing $d = 2$ in (4.11) and (4.12) we recover (2.14) and (2.15), respectively.

Taking the previous examples as a model, the reader can easily continue our calculations even with $l\left(H^d_N\right)$ and $l\left(P^t_M\right)$ for any $N, M \in \widehat{\mathbb{N}}$ and any finite integers $d, t \geq 2$.

5. Conclusions

The paper presents analyses and examples of computations resulting from the application of the Sergeyev's grossone-based system to Peano, Hilbert and Moore plane curves (P, H, $M \subset \mathbb{R}^2$, respectively), and moreover to Peano and Hilbert curves (P^d, $H^d \subset \mathbb{R}^d$, respectively) in any dimension $d \geq 2$. The main aim was to study, investigate, and explain the advantages emerging from thee use of the new framework in comparison with traditional computational techniques in the treatment of space-filling curves.

First of all, we have highlighted how the usual geometric process that builds a curve that fills a space by successive polygonal approximations, no longer generates, if seen in the new environment, a single limit curve but a whole family of infinitely many distinct curves, none of them fills the square $I^d \subset \mathbb{R}^d$ or the hypercube $I^d \subset \mathbb{R}^d$ in the new setting. Most of the examples and computations are originated from one-dimensional measures, i.e., finite and infinite lengths, while in Sect. 3 we have area measures related to Moore polygons and to the Moore curve. In particular, we have provided examples where objects, calculations, and results are completely different depending on whether we use the traditional framework or the new one (see (3.9) and (3.12)). Moreover, the series expansions achieved in (2.14), (2.15), (4.11), and (4.12) seem very promising in the context of the new theory and hopefully can lead to other interesting developments.

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References

[1] Alber, J., Niedermeier, R.: On multidimensional curves with Hilbert property. Theory Comput. Syst. **33**, 295–312 (2000). https://doi.org/10.1007/s002240010003

[2] Alber, J., Niedermeier, R.: On Multi-dimensional Hilbert Indexings, vol. 1449. Spinger LNCS, Berlin (1998)

[3] Antoniotti, L., Caldarola, F., d'Atri, G., Pellegrini, M.: New approaches to basic calculus: an experimentation via numerical computation. In: Sergeyev, Y.D., Kvasov, D.E. (eds) Proceedings of the 3rd International Conference NUMTA 2019 - Numerical Computations: Theory and Algorithms. Cham: Springer LNCS 11973 (2020), pp. 329–342. https://doi.org/10.1007/978-3-030-39081-5_29

[4] Baeder, M.: Space-Filling Curves: An Introduction with Applications in Scientific Computing. Text in computational science and engineering, vol. 9. Springer, Heidelberg (2013)

[5] Bertacchini, F., Bilotta, E., Caldarola, F., Pantano, P.: Complex interactions in one-dimensional cellular automata and linguistic constructions. Appl. Math. Sci. **12**(15), 691–721 (2018). https://doi.org/10.12988/ams.2018.8353

[6] Bertacchini, F., Bilotta, E., Caldarola, F., Pantano, P.: The role of computer simulations in learning analytic mechanics towards chaos theory: a course experimentation. Int. J. Math. Educ. Sci. Technol. **50**(1), 100–120 (2019). https://doi.org/10.1080/0020739X.2018.1478134

[7] Bertacchini, F., Bilotta, E., Caldarola, F., Pantano, P., Renteria Bustamante, L.: Emergence of Linguistic-like Structures in One-dimensional Cellular Automata. In: Sergeyev, Y.D., Kvasov, D.E., Dell'Accio, F., Mukhametzhanov, M.S. (eds) AIP proceedings of the 2nd International Conference on "Numerical Computations: Theory and Algorithms". New York: AIP Publ. 090044 (2016). https://doi.org/10.1063/1.4965408

[8] Butz, A.R.: Alternative algorithm for Hilbert's space filling curve. IEEE Trans. Comput. **20**, 424–426 (1971)

[9] Butz, A.R.: Convergence with Hilbert's space filling curve. J. Comput. Syst. Sci. **3**, 128–146 (1969)

[10] Butz, A.R.: Space filling curves and mathematical programming. Inf. Control **12**(4), 314–330 (1968)

[11] Caldarola, F.: The Sierpiński curve viewed by numerical computations with infinities and infinitesimals. Appl. Math. Comput. **318**, 321–328 (2018). https://doi.org/10.1016/j.amc.2017.06.024

[12] Caldarola, F.: The exact measures of the Sierpiński d-dimensional tetrahedron in connection with a Diophantine nonlinear system. Commun. Nonlinear Sci. Numer. Simul. **63**, 228–238 (2018). https://doi.org/10.1016/j.cnsns.2018.02.026

[13] Caldarola, F., Cortese, D., d'Atri, G., Maiolo, M.: Paradoxes of the infinite and ontological dilemmas between ancient philosophy and modern mathematical solutions. In: Sergeyev, Y.D., Kvasov, D.E. (eds) Proceedings of the 3rd International Conference "NUMTA 2019 - Numerical Computations: Theory and Algorithms". Cham: Springer LNCS 11973, pp. 358–372 (2020). https://doi.org/10.1007/978-3-030-39081-5_31

[14] Caldarola, F., Maiolo, M., Solferino, V.: A new approach to the Z-transform through infinite computation. Commun. Nonlinear Sci. Numer. Simul. **82**, 105019 (2020). https://doi.org/10.1016/j.cnsns.2019.105019

[15] Cococcioni, M., Pappalardo, M., Sergeyev, Y.D.: Lexicographic multi-objective linear programming using grossone methodology: theory and algorithm. Appl. Math. Comput. **318**, 298–311 (2018)

[16] D'Alotto, L.: A classification of one-dimensional cellular automata using infinite computations. Appl. Math. Comput. **255**, 15–24 (2015)

[17] D'Alotto, L.: Cellular automata using infinite computations. Appl. Math. Comput. **218**(16), 8077–82 (2012)

[18] De Leone, R.: The use of grossone in mathematical programming and operations research. Appl. Math. Comput. **218**(16), 8029–38 (2012)

[19] Haverkort, H.: An inventory of three-dimensional Hilbert space-filling curves. arXiv: 1109.2323v2 (2016)

[20] Haverkort, H.: How many three-dimensional Hilbert curves are there? arXiv: 1610.00155v2 (2017)

[21] Haverkort, H., van Walderveen, F.: Four-dimensional Hilbert curves for R-trees. ACM J. Exp. Alg. **16**, 34 (2011)

[22] Hilbert, D.: Über die stetige Abbildung einer Linie auf ein Flächenstück. Math. Ann. **38**(3), 459–460 (1891)

[23] Ingarozza, F., Adamo, M. T., Martino, M., Piscitelli, A.: A grossone-based numerical model for computations with infinity: a case study in an Italian high school. In: Sergeyev, Y.D., Kvasov, D.E. (eds) Proceedings of the 3rd International Conference on "NUMTA 2019 - Numerical Computations: Theory and Algorithms". Cham: Springer LNCS 11973, pp. 451–462 (2020). https://doi.org/10.1007/978-3-030-39081-5_39

[24] Lawder, A. K.: Calculation of mappings between one and n-dimensional values using the Hilbert space-filling curve. Research report JL1/00, School of Computer Science and Information Systems, Birkbeck College, Univ. London (2000)

[25] Li, C., Feng, Y.: Algorithm for Analyzing N-Dimensional Hilbert Curve. LNCS **3739**, 657–662 (2005)

[26] Margenstern, M.: Fibonacci words, hyperbolic tilings and grossone. Commun. Nonlin Sci. Numer. Simul. **21**(1–3), 3–11 (2015)

[27] Mazzia, F., Sergeyev, Y. D., Iavernaro, F., Amodio, P., Mukhametzhanov, M. S.: Numerical methods for solving ODEs on the Infinity Compute. In: Sergeyev, Y.D., Kvasov, D.E., Dell'Accio, F., Mukhametzhanov, M.S. (eds) AIP Proceedings of the 2nd International Conference on "Numerical Computations: Theory and Algorithms". New York: AIP Publ., 090033 (2016). https://doi.org/10.1063/1.4965397.

[28] Peano, G.: Sur une courbe qui remplit toute une aire plane. Math. Annaln. **36**, 157–160 (1980)

[29] Quinqueton, M.B.: A locally adaptive Peano scanning algorithm. IEEE Trans. PAMI **3**, 403–412 (1981)

[30] Rizza, D.: A study of mathematical determination through Bertrand's paradox. Philos. Math. **26**(3), 375–395 (2018)

[31] Rizza, D.: Primi passi nell'aritmetica dell'infinito. Preprint (2019) (in Italian)

[32] Rizza, D.: Supertasks and numeral system. In: Sergeyev, Y.D., Kvasov, D.E., Dell'Accio, F., Mukhametzhanov, M.S. (eds) AIP Proceedings of the 2nd International Conference on "Numerical Computations: Theory and Algorithms". New York: AIP Publ. 090005 (2016). https://doi.org/10.1063/1.4965369.

[33] Sagan, H.: A three dimensional Hilbert curve. Int. J. Math. Educ. Sci. Technol. **24**(4), 541–545 (1993)

[34] Sagan, H.: Space-Filling Curves. Springer, New York (1994)

[35] Sergeyev, Y.D.: A new applied approach for executing computations with infinite and infinitesimal quantities. Informatica **19**(4), 567–96 (2008)

[36] Sergeyev, Y.D.: Arithmetic of infinity. 2nd Electronic Ed. 2013. Cosenza: Edizioni Orizzonti Meridionali (2003)

[37] Sergeyev, Y.D.: Blinking fractals and their quantitative analysis using infinite and infinitesimal numbers. Chaos Solitons Fractals **33**(1), 50–75 (2007)

[38] Sergeyev, Y.D.: Computations with grossone-based infinities. In: Calude, C.S., Dinneen, M.J. (eds) Unconventional Computation and Natural Computation: Proceedings of the 14th International Conference UCNC 2015. New York: Springer, LNCS 9252, pp. 89–106 (2015)

[39] Sergeyev, Y.D.: Evaluating the exact infinitesimal values of area of Sierpinskinski's carpet and volume of Menger's sponge. Chaos Solitons Fractals **42**(5), 3042–6 (2009)

[40] Sergeyev, Y.D.: Higher order numerical differentiation on the Infinity Computer. Optim. Lett. **5**(4), 575–85 (2011)

[41] Sergeyev, Y.D.: Lagrange lecture: methodology of numerical computations with infinities and infinitesimals. Rend. Semin. Mat. Univ. Polit. Torino **68**(2), 95–113 (2010)

[42] Sergeyev, Y.D.: Numerical infinities and infinitesimals: methodology, applications, and repercussions on two Hilbert problems. EMS Surv. Math. Sci. **4**(2), 219–320 (2017)

[43] Sergeyev, Y.D.: Solving ordinary differential equations by working with infinitesimals numerically on the infinity Computer. Appl. Math. Comput. **219**(22), 10668–81 (2013)

[44] Sergeyev, Y.D.: The exact (up to infinitesimals) infinite perimeter of the Koch snowflake and its finite area. Commun. Nonlinear Sci. Numer. Simul. **31**, 21–29 (2016)

[45] Sergeyev, Y.D.: Un semplice modo per trattare le grandezze infinite ed infinitesime. Mat. Cultura Soc. Riv. Unione Mat. Ital. **8**(1), 111–47 (2015) (in Italian)

[46] Sergeyev, Y.D.: Using blinking fractals for mathematical modelling of processes of growth in biological systems. Informatica **22**(4), 559–76 (2011)

[47] Sergeyev, Y.D., Garro, A.: Observability of Turing machines: a refinement of the theory of computation. Informatica **21**(3), 425–54 (2010)

[48] Sergeyev, Y.D., Mukhametzhanov, M.S., Mazzia, F., Iavernaro, F., Amodio, P.: Numerical methods for solving initial value problems on the Infinity Computer. Int. J. Unconvention. Comput. **12**(1), 55–66 (2016)

[49] Sergeyev, Y.D., Strongin, R.G., Lera, D.: Introduction to Global Optimization Exploiting Space-Filling Curves. Springer, New York (2013)

[50] Spanier, E.H.: Algebraic Topology. 3rd corr. Print. Springer, New York (2012)

[51] Young, W.H., Young, G.C.: The Theory of Sets of Points. 2nd Edition, 2013, Volume 259 of Chelsea Publishing Series, AMS (1972)

[52] Zhigljavsky, A.: Computing sums of conditionally convergent and divergent series using the concept of grossone. Appl. Math. Comput. **218**(16), 8064–76 (2012)

[53] Zilinskas, A.: On strong homogeneity of two global optimization algorithms based on statistical models of multimodal objective functions. Appl. Math. Comput. **218**(16), 8131–36 (2012)

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