New approaches to basic calculus: an experimentation via numerical computation

Luigi Antoniotti, Fabio Caldarola[0000−0002−4011−848X], Gianfranco d'Atri, and Marco Pellegrini[0000−0002−5635−3097]

Dep. of Mathematics and Computer Science, Cubo 31/A,
Università della Calabria, Arcavacata di Rende 87036 (CS), Italy
antoniottiluigi@tiscali.it, caldarola@mat.unical.it, datri@mat.unical.it, m19700530@hotmail.com

Abstract. The introduction of the first elements of calculus both in the first university year and in the last class of high schools, presents many problems both in Italy and abroad. Emblematic are the (numerous) cases in which students decide to change their course of study or give it up completely cause the difficulties with the first exam of mathematics, which usually deals with basic calculus. This work concerns an educational experimentation involving (with differentiated methods) about 170 students, part at the IPS "F. Besta" in Treviso (IT) with main focus on two 5th classes where the students' age is about 19 years old, and part at the Liceo Classico Scientifico "XXV Aprile" in Pontedera, prov. of Pisa (IT). The experimental project aims to explore the teaching potential offered by non-classical approaches to calculus jointly with the so-called "unimaginable numbers". In particular, we employed the computational method recently proposed by Y.D. Sergeyev and widely used both in mathematics, in applied sciences and, recently, also for educational purposes. In the paper will be illustrated tools, investigation methodologies, collected data (before and after the teaching unit), and the results of various class tests.

Keywords: Mathematical education · Learning/teaching models and strategies · Interactive learning environments · Infinite · Infinity Computer · Grossone · Unimaginable numbers

1 Introduction

The difficulties that high school students encounter in their approach to university mathematics courses are fairly well known, both because of the direct experience of those who teach them, and because of the extensive literature that deals with various aspects of the problem. For example, see [19] for the intrinsic difficulties related to the transition to the third level of education, or [25, 27–30] for detailed analyses of students' problems and approaches to calculus courses. A research topic, very lively as well, which often overlaps with the previous ones, deals with the role assumed by software, simulations and computer technologies in general, as a support to academic teaching, and to the learning and personal elaboration of concepts and methods by the students (see, e.g., [24, 45]). In this view, very interesting results were obtained in the paper [4], which in part served as a model to build the working environments described here.[1]

A whole line of research, within the theme concerning the students' approach to calculus, focuses on methods and problems related to teaching and learning the concept of limit and the processes with infinitely many steps. For example, [14, 15, 18, 30, 44] investigate student's difficulties, common misconceptions and obstacles for the learning, [49, 50] elaborate on conceptual images and students' models of limit, and the very interesting paper [47] deals with the mathematical intuition in limiting processes.[2]

We are interested, instead, in the interactions of the students, in particular of the last year of high schools, with the first rudiments of non-classical analysis systems and similar mathematics. The teaching of non-classical mathematics, in many countries of the world, has become a consolidated fact from many years; for example, the broader phenomenon is the teaching of Robinson's *non-standard analysis* both in university courses[3] and in high schools. In Italy, in general, traditional methods are taught almost everywhere, but the focus on non-classical mathematics is strongly growing and many people are convinced of the usefulness of teaching alternative theories and methods also in high schools.[4]

The research exposed in this paper wants to investigates the response of students with respect the easy computational system proposed by Y. Sergeyev and, in smaller measure, the notational system introduced by D.E. Knuth to write the so-called *unimaginable numbers* (see Section 2 for a brief theoretical overview). The experimentations described here involve 8 high school classes in two different Italian institutes in Treviso and Pontedera (Pi), see Section 3 for details. The aim of the research are multiple, and can be summarized in three main groups/points to investigate:

[1] The paper [4] concerns a rather complex two-year experimental project conducted at the University of Calabria within the Master's Degree Program in Electronic Engineering. There, once two groups were formed each year, an experimental one equipped with computer-based tools and a control group associated with more traditional teaching methods, the aim was to analyze a series of student performances. We inform the reader that in the experimentations described in the present paper, we will find some slight traces of part of the methods used in [4]; the most visible is the employ of the computational system called *Infinity Computer* which will be used by a (very small) part of the sample, creating a hint of parallelism with the role of the software used in [4].

[2] The mathematical intuition will be important also for us, when our students will work with the infinite and, in particular, when they will approach the new concept of *grossone* (see also [46]).

[3] See, for instance, the manual [23] adopted in many academic courses and now in its third edition.

[4] See, for instance, [16, pag. 2] and the proceedings [7] of the national Italian conference "Analisi nonstandard per le scuole superiori. VII Giornata di studio". This conference takes place every year and has now reached its ninth edition, Verona, October 5, 2019.

(a) The students' mathematical intuition and first approaches with respect the grossone system without preliminary class lectures;
(b) The students' responses, empathy and performances during and after a brief cycle of class lectures;
(c) The students' individual home working with also the support of the Infinity Computer.

In Section 4 we will give the results and some brief conclusions.

One last important piece of information for the reader: the paper [22] concerns a twin but independent experimentation, concerning the same themes and carried out approximately simultaneously.

2 A brief overview on the grossone-based system and on unimaginable numbers

Sergeyev began to develop the grossone-based methodology in the early 2000's and, in brief, we can say that his numerical system is based on two fundamental units: the familiar 1 that generates natural numbers and a new infinite unit ①, called *grossone*, which allows to write infinite numbers and to execute computations with them in a *intuitive* and easy way, similar at all to what people daily do with finite numbers and ordinary integers. Moreover, infinitesimal numbers then appear by taking the inverses of infinite, grossone-based numbers. We also recall that, in Sergeyev's system, ① is the number of elements of $\mathbb{N}$ and $n \leq$ ① for all $n \in \mathbb{N}$.

The *extended set of natural numbers*, denoted by $\widehat{\mathbb{N}}$, can be written as

$$\widehat{\mathbb{N}} = \Big\{ \underbrace{1, 2, \ldots, ① - 1, ①}_{\mathbb{N}}, ① + 1, \ldots, 2①, \ldots, 3①, \ldots,$$
$$①^2, ①^2 + 1, \ldots, ①^2 + ①, \ldots, 2①^2, \ldots, ①^3, \ldots, 2^{①}, \ldots,$$
$$2^{①} + ①, \ldots\ldots, ①^{①}, \ldots\ldots \Big\},$$

while the *extended set of integers* as $\widehat{\mathbb{Z}} = \widehat{\mathbb{N}} \cup \{0\} \cup \{-N : N \in \widehat{\mathbb{N}}\}$. Hence, the quotients of elements of $\widehat{\mathbb{Z}}$ obviously yield the *extended set of rational numbers* denoted with $\widehat{\mathbb{Q}}$.

Introductory books for the grossone-based numerical system are [31, 33], written in popular or didactic way, while the interested reader can see [32, 37, 39, 41] for more detailed surveys and [1, 8–10, 13, 34, 35, 38, 39, 41, 42] (and the references therein) for some applications. Sergeyev's *Infinity Computer* [36] is, moreover, an on-line computing software developed for the grossone-based system which gives an extra level to our experimentation and connects it to previous works like [2, 4] and to other similar ones in progress using the frameworks of [3, 5].

"Unimaginable numbers" are instead finite but very large natural numbers with a completely different and ancient origin: the first unimaginable number comes back in fact to Archimedes of Syracuse (see [6, 11]) Usually an integer $n \in \mathbb{N}$ is said *unimaginable* if it is greater than 1 *googol* which is equal to 10^{100}. Writing unimaginable numbers in common scientific notation is almost always impossible and we need notations developed *ad hoc* like Knuth's up-arrow notation that at its lower kevels gives the usual addition, multiplication and exponentiation, then tetration, pentation, hexation, and so on (see [6, 11, 12, 26] and the references therein).

3 Description of the samples and the methods of the research

The participants to this research are divided in more groups:

(P) A group P of students at the *Liceo Classico Scientifico "XXV Aprile"* in Pontedera, prov. of Pisa, Italy. This group consists of 45 students coming from a second and a third class named, by convenience, $P2$ and $P3$, respectively. More details on this sample will be given in Table 1.

(T) A group T of students at the *Istituto Superiore Commerciale Professionale "F. Besta"* in Treviso, Italy. This group comes from 2 third and 2 fourth classes conventionally indicated by $T3$, $T3'$, $T4$ and $T4'$, respectively.

($\mathcal{T}$) A second group of students at the same institute in Treviso as group T, is now indicated by the uppercase calligraphic letter $\mathcal{T}$: such a group comes from 2 fifth classes conventionally indicated by $\mathcal{T}5$ and $\mathcal{T}5'$. This group is the more interesting both for being the last class before the university career (for those who will choose to continue their studies) and because the organization, started before for this group, allowed to carry out a more extensive and articulated research.

A more accurate description of the groups listed above can be found in Table 1 where some data are shown for each class of the samples P, T and $\mathcal{T}$. The second column of Table 1 enumerates the students in each class and the third one gives the subdivision male-female. The 4th column lists the "mean age" computed at April 1, 2019 (in some case there is a certain margin of approximation caused by incomplete data). The last three columns regard the final votes obtained in mathematics at the end of the first semester (January 31, 2019): the 5th column computes the "mean vote" of all the students, the 6th one gives separated means male-female, and the last provides the maximum final vote in the class, separately again male-female.

3.1 The group P: experimental activities and methods

For the group P the experimental activity was limited to a class test at the end of May 2019, in agreement with the aim (a) described in the Introduction. The class test was administered without prior knowledge of the grossone-based system and without students have seen before the symbol ① (in brief, a "zero-knowledge test"). The only information given to them was ① $\in \mathbb{N}$ and ① $\geq n$ for

Table 1. The composition of the 8 classes constituting the samples, and some other data.

Class	Students	M - F	Mean age	Mean vote	Mean vote (M-F)	Max vote (M-F)
$P2$	25	11 - 14	15.2	6.7/10	6.5/10 - 6.8/10	9/10 - 9/10
$P3$	20	10 - 10	16.2	7.6/10	7.9/10 - 7.3/10	9/10 - 8/10
$T3$	26	8 - 18	16.6	5.6/10	5.7/10 - 5.6/10	8/10 - 9/10
$T3'$	17	6 - 11	16.6	5.8/10	5.9/10 - 5.7/10	9/10 - 9/10
$T4$	15	4 - 11	17.5	5.8/10	5.9/10 - 5.7/10	8/10 - 9/10
$T4'$	27	8 - 19	17.7	5.4/10	5.6/10 - 5.0/10	8/10 - 9/10
$\mathcal{T}5$	23	7 - 16	18.7	5.5/10	6.1/10 - 5.2/10	9/10 - 9/10
$\mathcal{T}5'$	15	4 - 11	19.1	5.6/10	5.8/10 - 5.5/10	9/10 - 8/10

all $n \in \mathbb{N}$. The test had 15 multiple-choice questions and other 6 open ones. The contents were elementary operations with ① and about the order relation in $\widehat{\mathbb{Q}}$, i.e., the *extended set of rational numbers* in Sergeyev's framework (see [32, 33, 37, 39, 41]). Below we report in English some examples of the proposed questions.

Question 1.[5] Consider the writing ①+① and make a mark on the option that seems most correct to you among the following:

(a) ① + ① has no sense;
(b) ① + ① = ①;
(c) ① + ① is impossible to execute;
(d) ① + ① = 2①;
(e) ① + ① = 0.

Question 2. Consider the writing ①−① and make a mark on the option that seems most correct to you among the following:

(a) ① − ① = −①;
(b) ① − ① is indeterminate;
(c) ① − ① = 0;
(d) ① − ① = ①;
(e) ① − ① has no sense.

Question 3. Consider the expression −3①+① and make a mark on the option that seems most correct to you among the following:

(a) −3① + ① = ①;
(b) −3① + ① is a writing without sense;
(c) −3① + ① = −3①;

[5] The numbers used here to enumerate questions are different from the those in the students's test (cf. [22, Section 3]). We moreover precise that in some classes in Trento we prepared two or four test versions changing for the order of questions and answers, to prevent, together with other appropriate measures, any kind of influence among students.

(d) $-3① + ① = -2①$;
(e) $-3① + ① = ①$;
(f) $-3① + ①$ is an indeterminate expression;
(g) $-3① + ① = 0$.

Question 4. Consider ①, $① + ①$ and $① \cdot ①$: mark the option (or the options) that seem correct to you among those below.

(a) $① < ① + ①$, but $① \cdot ① = ①$;
(b) $① < ① + ① < ① \cdot ①$;
(c) $① + ①$ and $① \cdot ①$ are both equal to ①;
(d) $① \leq ① + ① \leq ① \cdot ①$;
(e) It is not possible to establish any order relation between ①, $① + ①$ and $① \cdot ①$;
(f) $① < ① + ①$ and $① + ① \leq ① \cdot ①$;
(g) $① \leq ① \cdot ① < ① + ①$;
(h) $① \leq ① \cdot ① \leq ① + ①$;
(i) The writings $① \cdot ①$ and $① + ①$ have no sense;
(j) None of the previous is correct;
(k) Other: ____

Question 5. Consider the expression $-\frac{5}{3}① + \frac{1}{2}①$ and choose the option that seems most correct to you among the following:

(a) Both $-\frac{5}{3}①$ and $\frac{1}{2}①$ are writings without sense;
(b) $-\frac{5}{3}① + \frac{1}{2}①$ is an indeterminate expression;
(c) $-\frac{5}{3}① + \frac{1}{2}① = -①$;
(d) $-\frac{5}{3}① + \frac{1}{2}① = -\frac{7}{6}①$;
(e) $-\frac{5}{3}① + \frac{1}{2}① = 0$;
(f) $-\frac{5}{3}① + \frac{1}{2}① = ①$.

Question 6. Consider the expression $\left(-\frac{2}{3}① + 2\right) \cdot (4① - 3) + 4$ and choose the option that seems most correct to you among the following:

(a) The first factor is equal to $-①$, the second to $+①$ and the addition of 4 is irrelevant, hence $\left(-\frac{2}{3}① + 2\right) \cdot (4① - 3) + 4 = -①$;
(b) $\left(-\frac{2}{3}① + 2\right) \cdot (4① - 3) + 4 = -①^2$;
(c) $\left(-\frac{2}{3}① + 2\right) \cdot (4① - 3) + 4$ is an indeterminate expression;
(d) $\left(-\frac{2}{3}① + 2\right) \cdot (4① - 3) + 4 = -\frac{8}{3}①^2 + 10① - 2$;
(e) It is not possible to sum $\frac{2}{3}①$ with 2 and $4①$ with -3, hence the expression $\left(-\frac{2}{3}① + 2\right) \cdot (4① - 3) + 4 = -①$ has no sense.

Question 7. Consider $\frac{12}{8}①$ and $\frac{5}{3}①$: mark the option (or the options) that seem correct to you among those below.

(a) The writings $\frac{12}{8}①$ and $\frac{5}{3}①$ have no sense;

(b) $\frac{12}{8}①=\frac{5}{3}①$;
(c) $\frac{12}{8}①$ and $\frac{5}{3}①$ are both equal to ①;
(d) $\frac{12}{8}①>\frac{5}{3}①$;
(e) $\frac{12}{8}①<\frac{5}{3}①$;
(f) $\frac{12}{8}①\geq\frac{5}{3}①$;
(g) $\frac{12}{8}①\leq\frac{5}{3}①$;
(h) It is not possible to establish an order relation between $\frac{12}{8}①$ and $\frac{5}{3}①$.

It is important to notice that the questions in the test (and also for the ones of the samples T and $\mathcal{T}$) were grouped in small groups on separate sheets, and the students were asked to read and answer the questions in the order of presentation, without the possibility of changing an answer already given.

After the day of the test, and therefore outside our experimentation, there were some discussions in classroom, often to answer the questions and curiosities of the students themselves, about the meaning and use of the symbol ①. We specify, however, that the experimentation in question did not in any way affect the regular progress of the established school program.

The results of the test will be given and analyzed in Section 4.

3.2 The group T: experimental activities and methods

The experimental activities and the research methodologies used for the 4 classes of group T were almost identical to those described in Subsection 3.1. The only exceptions concern a greater ease and a simplification of some test questions, especially of some more complex ones not reported in the previous subsection, dictated by the different type of school and, unlike the group P, to classes $T3$ and $T4'$ were administered, in the days after the test, an informative questionnaire that aimed to capture the (eventual) interest and any curiosity aroused in the students. We also precise that the test questions and their number were not the same for all the classes that made up the group T, for didactic and organizational reasons.

3.3 The group $\mathcal{T}$: experimental activities and methods

The most complex and structured experimentation concerned the group $\mathcal{T}$. In addition to the zero-knowledge test as for groups P and T, but carried out about one or two months in advance, a series of short lessons and class discussions was proposed to both classes $\mathcal{T}5$ and $\mathcal{T}5'$. The organization relative to the class $\mathcal{T}5$ allowed also the possibility of a greater number of lessons, which in any case covered a very limited amount of time (about 4 or 5 hours split into packages of half an hour each time, over two or three months). After the cycle of lectures, a test very similar, especially for the class $\mathcal{T}5'$, to the one administered at the beginning was proposed for both classes: for convenience and future references we call it the *final test*.

Our experimental activity with group $\mathcal{T}$ included also some rudiments of calculus. In particular, it has been discussed the concept of limit for $x \to \pm\infty$ in classical analysis compared with the evaluation of a function at a grossone-based infinity in Sergeyev's framework (mainly we took $x = \pm①$ than other infinite numbers). Moreover, we showed the relation between some asymptotic behaviours of a function and its derivative both in the traditional and in the new context, and we also talked about the meaning and the way to perform computations with infinitesimal quantities written in the new system. Examples of closed and open questions proposed to the students of group $\mathcal{T}$ in the final test are reported below translated in English.

Question 8. Consider the writing $\frac{3}{2①} - \frac{1}{①}$ and mark the right option/options among the following:

(a) $\frac{3}{2①} - \frac{1}{①} = \frac{1}{2}①$;
(b) $\frac{3}{2①} - \frac{1}{①}$ is indeterminate;
(c) The writings $\frac{3}{2①}$ and $\frac{1}{①}$ have no sense because it is not possible to divide by infinity;
(d) $\frac{3}{2①} - \frac{1}{①}$ is zero because both $\frac{3}{2①}$ and $\frac{1}{①}$ are equal to zero;
(e) $\frac{3}{2①} - \frac{1}{①} = \frac{1}{2①}$;
(f) $\frac{3}{2①} - \frac{1}{①} = \frac{2}{①}$;
(g) $\frac{3}{2①} - \frac{1}{①} = \frac{3-1}{2①} = \frac{1}{①}$;
(h) $\frac{3}{2①} - \frac{1}{①} = 0.5①^{-1}$.

Question 9. Consider the writings $\frac{1}{2①}$, $-\frac{1}{①}$ and $-3\frac{1}{2①} + \frac{1}{2}$. Indicate the true expressions among the following:

(a) $-\frac{3}{2①} + \frac{1}{2} < -\frac{1}{①} < \frac{1}{2①}$;
(b) $-\frac{1}{①} < -\frac{3}{2①} + \frac{1}{2} < \frac{1}{2①}$;
(c) $-\frac{1}{①} < \frac{1}{2①} < -\frac{3}{2①} + \frac{1}{2}$;
(d) There are no order relations between $\frac{1}{2①}$, $-\frac{1}{①}$ and $-3\frac{1}{2①} + \frac{1}{2}$ because they are not numbers;
(e) $-\frac{1}{①} = \frac{1}{2①} = 0 < -\frac{3}{2①} + \frac{1}{2} = \frac{1}{2}$;
(f) The expressions $\frac{1}{2①}$, $-\frac{1}{①}$ and $-3\frac{1}{2①} + \frac{1}{2}$ have no sense because it is not possible to divide by infinity.

Other questions of the test had the aim to solicit a comparison with the symbol ∞ as used in traditional mathematics.

Question 10. Consider the writing $\infty + \infty$ and $\infty \cdot \infty$. Choose the true options among the following:

(a) $\infty + \infty$ and $\infty \cdot \infty$ have no sense in mathematics;
(b) $\infty + \infty = \infty$ and $\infty \cdot \infty = \infty$;
(c) $\infty + \infty = \infty$ and $\infty + \infty = \infty$;
(d) $\infty + \infty = 2\infty < \infty \cdot \infty = \infty^2$;
(e) $\infty + \infty$ and $\infty \cdot \infty$ are not comparable;
(f) $\infty < \infty + \infty < \infty \cdot \infty$.

Question 11. Consider the writing $\infty - \infty$ and mark the right options among the following:

(a) $\infty - \infty = \infty$;
(b) $-\infty < \infty - \infty < \infty$;
(c) $\infty - \infty$ is an indeterminate expression;
(d) $\infty - \infty = 0$;
(e) $-\infty < \infty - \infty < \infty$;
(f) $\infty - \infty < \infty + \infty$;
(g) $\infty - \infty$ and $-\infty$ are not comparable.

Question 12. For each of the following items make a mark on "T" or "F" if you believe the corresponding statement to be true or false, respectively.

(a)	$① < +\infty$	T - F
(b)	$① = +\infty$	T - F
(c)	① and ∞ are not comparable	T - F
(d)	$① \leq +\infty$	T - F
(e)	$① \geq +\infty$	T - F
(f)	① and ∞ cannot be used together because the belong to different settings	T - F
(g)	$①^2 > +\infty$	T - F
(h)	$① + 1 = ①$	T - F
(i)	$\infty + 1 = \infty$	T - F
(j)	$\infty + 1 > \infty$	T - F

Question 13. In the classical setting, consider the function given by the analytical expression $f(x) = \frac{x^2}{x+1}$.

(a) Compute the domain of the function and the limits at the extremal points of the domain.
(b) Compute the asymptotes of f (vertical, horizontal and oblique).

Question 14. In the grossone setting, consider the function $f(x) = \frac{x^2}{x+1}$.

(a) Compute the values of f at $-①$ and $①$.
(b) Compute the values of f at $-① + 2$, $-① - 2$ and $① + 1$.
(c) Let $y = a(x)$ be the right oblique asymptote of f (if it exists).
   - Compute the value $a(①)$ and $a(① + 1)$.
   - Compute the difference $f(①) - a(①)$ and $f(① + 1) - a(① + 1)$.

Question 15 (excluded from the evaluation). In your opinion there are some advantages in using ① in the place of $+\infty$ (i.e., the grossone setting in the place of the classical one)? Justify the answer and, if yes, list some of them.

The last extra question, although proposed together the final test, was excluded from any attribution of a score and this was clearly written. In any case, most of the students did not answer this question, or gave hasty and little significant answers: a different outcome would probably have been recorded if it had been proposed on a different day rather than at the end of a test with many questions.

The students of class $\mathcal{T}5$ have also been spoken in classroom of the Infinity Computer, trying to motivate them in a deepening and individual work at home on it. From the compilation of the informative questionnaire and from a continuous dialogue with the students, however, it emerged that only 4 of them actually used the Infinity Computer at least once at home, independently. For convenience, we will denote this group of students by $\mathcal{T}5.1$.

As regards unimaginable numbers we inform the reader that, for the class $\mathcal{T}5$, a soft approach to Knuth's notation and very large numbers was also planned, but just in part developed with the students. In particular we presented tetrations and pentations to them, and in a first moment it seemed very successful, in particular the way to iterate the ordinary exponentiation and to write it compactly under the form of a tetration. Many problems and much confusion emerged instead in a successive lesson two weeks later, and we decided to ask no questions about these topics in the final test (cf. the conclusions in Section 5).

4 Results and their analysis

In this section we will give the results of the tests, will discuss the situation picture emerging from the questionnaires and the dialogues with the students, and will finally draw up a balance of the experimentations giving some brief conclusions jointly with the next section.

The zero-knowledge test has been proposed to all the 8 classes, with major differences between the groups P and $T \cup \mathcal{T}$, but with several minor ones inside the sample $T \cup \mathcal{T}$. In Table 2, the columns 2-5 are devoted to this test: the 2nd and 3rd columns give the mean student score [Mean] obtained in each class and the maximum score [Max], respectively, both over the total score of the test. The 4th column gives the normalized mean (students') score [NMS] and the 5th column the standard deviation [SD]. The column from 6 to 9 are the twin ones of 2-5, but relative to the final test (notations are hence obtained by adding "$\mathcal{F}$" to the previous ones).

The results of the zero-knowledge test are, in general, positive or very positive, in dependence of the cases, as Table 2 shows. In particular, the maximum scores in the 3rd column unequivocally give the measure of how intuitive it is to perform basic calculations with the grossone system.

The scores of the final test are higher and with more gap from the corresponding one of the initial test, in particular for the classes with less teaching

Table 2. The results of the tests for each class.

Class	Mean	Max	NMS	SD	Mean $\mathcal{F}$	Max $\mathcal{F}$	NMS $\mathcal{F}$	SD $\mathcal{F}$
$P2$	15.9/21	21/21	0.76	2.7/21	-	-	-	-
$P3$	16.7/21	20/21	0.79	2.1/21	-	-	-	-
$T3$	10.5/24	18/24	0.43	2.9/24	15.1/24	21/24	0.63	2.7/24
$T3'$	8.1/24	15/24	0.34	3.9/24	-	-	-	-
$T4$	10.8/24	15/24	0.45	4.1/24	-	-	-	-
$T4'$	8.4/24	15/24	0.35	3.7/24	13.2/24	20/24	0.55	3.1/24
$\mathcal{T}5$	16.2/24	24/24	0.67	5.5/24	20.7/24	24/24	0.86	2.7/24
$\mathcal{T}5'$	8.1/24	18/24	0.34	3.9/24	14.2/24	22/24	0.59	3.6/24
$\mathcal{T}5.1$	21.8/24	24/24	0.91	1.3/24	23.3/24	24/24	0.97	1.3/24

time. This result is rather unexpected and seems to be due to the fact that the class $\mathcal{T}5$, the one with the highest number of lessons, starts from a very high score in the initial test. But it could also mean that a few quick explanations are enough to significantly improve some good initial performances (considering that it is a text with zero knowledge) especially if not very high.

As regards the small $\mathcal{T}5.1$ group (i.e., the one consisting of the 4 students of the class $\mathcal{T}5$ that used the Infinity Computer at least once at home), we can observe very high performances in both columns 2 and 6, with a small increase. Probably, a more difficult extra test for this group would have been interesting both at the beginning and at the end of the experimentation. Finally, from the questionnaire and, in particular, from a continuous conversation with the students, we think that their approach with the Infinity Computer has been fruitful to arouse attraction and to give further motivation and interest, probably because it is seen as a form of concrete application (recall that we are dealing with a technical-commercial school) of the grossone-based system.

5 Conlusion

The good results obtained in the various levels of the experimentation have shown a remarkable usability for the students of the new concept of infinity represented by grossone. It should in fact be emphasized that in all the 8 classes that took part in the experimentation, most of the students succeeded in assimilating, or better, effectively understanding, the distinctive properties of ① and the correct way of performing calculations in the associate numerical-computational system in a few minutes, already at the initial zero-knowledge test. Very interesting and useful to the students, it was also the comparison, made several times during the lessons given to the classes $\mathcal{T}5$ and $\mathcal{T}5'$, between the classical conception of infinity (dating back to Cantor and Weierstrass) and that related to grossone: in fact the students, stimulated by the possibility of working computationally with ① in an "unconsciously familiar" way, showed a marked interest, very difficult to be aroused in general, also for the more theoretical aspects concerning the two models of infinity. The possibility of carrying out a wider experimentation and proposing Sergeyev's model on a larger scale could therefore have relevant educational implications.

We also believe that the Infinity Computer can also have a good educational value, which however we have not had the opportunity to investigate or test in depth in a very short experimentation, and with many new features for students like ours.

Similarly, with regard to unimaginable numbers, we conclude that they could have very interesting didactic applications (especially from the point of view of generalizing the usual operations of addition, multiplication and exponentiation via tetration, pentation, hexation, etc.), but they require experimental activities completely dedicated to them because a not so easy assimilation of such topics has emerged, at least among the students of a technical school, more used to calculations than to algebraic formalisms.

Aknowledgments

This work is partially supported by the research projects *"IoT&B, Internet of Things and Blockchain"*, CUP J48C17000230006, POR Calabria FESR-FSE 2014-2020.

References

1. Antoniotti, L, Caldarola, F., Maiolo, M.: Infinite numerical computing applied to Peano's, Hilbert's and Moore's curves. To appear in Mediterranean Journal of Mathematics.
2. Bertacchini, F., Bilotta, E., Pantano, P., Tavernise, A.: Motivating the learning of science topics in secondary schools: a constructivistic edutainment setting for studing Chaos. Computer and Education **59**(4), 1377–1386 (2012)
3. Bertacchini, F., Bilotta, E., Caldarola, F., Pantano, P.: Complex Interactions in One-dimensional Cellular Automata and Linguistic Constructions. Applied Mathematical Sciences **12**(15), 691–721 (2018). https://doi.org/10.12988/ams.2018.8353
4. Bertacchini, F., Bilotta, E., Caldarola, F., Pantano, P.: The Role of Computer Simulations in Learning Analytic Mechanics towards Chaos Theory: a Course Experimentation. International Journal of Mathematical Education in Science and Technology **50**(1), 100–120 (2019). https://doi.org/10.1080/0020739X.2018.1478134
5. Bertacchini, F., Bilotta, E., Caldarola, F., Pantano, P., Renteria Bustamante, L.: Emergence of Linguistic-like Structures in One-dimensional Cellular Automata. In: Sergeyev, Ya.D., Kvasov, D.E., Dell'Accio, F., Mukhametzhanov, M.S. (eds.) 2nd Int. Conf. "NUMTA 2016 - Numerical Computations: Theory and Algorithms", AIP Conf. Proc., vol. 1776, 090044. AIP Publ., New York (2016). https://doi.org/10.1063/1.4965408
6. Blakley, G.R., Borosh, I.: Knuth's iterated powers. Advances in Mathematics **34**(2), 109–136 (1979). https://doi.org/10.1016/0001-8708(79)90052-5
7. Bonavoglia, P. (ed.): Atti del convegno della VII Giornata Nazionale di Analisi non Standard per le scuole superiori, Venezia, 30 settembre 2017. Edizioni Aracne, Roma (2017) (in Italian)

8. Caldarola, F.: The Sierpiński curve viewed by numerical computations with infinities and infinitesimals. Applied Mathematics and Computation **318**, 321–328 (2018). https://doi.org/10.1016/j.amc.2017.06.024
9. Caldarola, F.: The exact measures of the Sierpiński d-dimensional tetrahedron in connection with a Diophantine nonlinear system. Communications in Nonlinear Science and Numerical Simulation **63**, 228–238 (2018). https://doi.org/10.10.16/j.cnsns.2018.02.026
10. Caldarola, F., Cortese, D., d'Atri, G., Maiolo, M.: Paradoxes of the infinite and ontological dilemmas between ancient philosophy and modern mathematical solutions. In: Sergeyev, Y.D., Kvasov, D.E. (eds.) 3rd Int. Conf. "NUMTA 2019 - Numerical Computations: Theory and Algorithms", LNCS, vol. 11973, pp. 358–372. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-39081-5_31
11. Caldarola, F., d'Atri, G., Maiolo, M.: What are the unimaginable numbers? Submitted for publication.
12. Caldarola, F., d'Atri, G., Mercuri, P., Talamanca, V.: On the arithmetic of Knuth's powers and some computational results about their density. In: Sergeyev, Y.D., Kvasov, D.E. (eds.) 3rd Int. Conf. "NUMTA 2019 - Numerical Computations: Theory and Algorithms", LNCS, vol. 11973, pp. 381–388. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-39081-5_33
13. Caldarola, F., Maiolo, M., Solferino, V.: A new approach to the Z-transform through infinite computation. Communications in Nonlinear Science and Numerical Simulation **82**, 105019 (2020). https://doi.org/10.1016/j.cnsns.2019.105019
14. Cornu, B.: Quelques obstacles à l'apprentissage de la notion de limite. Recherches en Didactique des Mathematiques **4**, 236–268 (1983)
15. Davis, R.B., Vinner, S.: The notion of limit: Some seemingly unavoidable misconception stages. The Journal of Mathematical Behavior **5**, 281–303 (1986)
16. Di Nasso, M.: Un'introduzione all'analisi con infinitesimi. In "Analisi nonstandard per le scuole superiori - V Giornata di studio", Atti del convegno, Verona, 10 Ottobre 2015, Matematicamente.it (2016) (in Italian)
17. Ely, R.: Nonstandard Student Conceptions About Infinitesimals. Journal for Research in Mathematics Education **41**(2), 117–146 (2010)
18. Ervynck, G.: Conceptual difficulties for first year university students in the acquisition of the notion of limit of a function. Proceedings of the Fifth Conference of the International Group for the Psychology of Mathematics Education, 330–333 (1981)
19. Guzman, M., Hodgson, B., Robert, A., Villani, V.: Difficulties in the passage from secondary to tertiary education. In: Fischer, G., Rehmann, U., (eds.) Proceedings of the International Congress of Mathematicians, Berlin 1998. Vol. 3, pp. 747–762 (1998)
20. Holton, D. (ed.): The Teaching and Learning of Mathematics at University Level, an ICMI Study. Kluwer Academic (2001)
21. Iannone P., Rizza D., Thoma A.: Investigating secondary school students' epistemologies through a class activity concerning infinity. In: Bergqvist E. et al. (eds.) Proceedings of the *42nd Conference of the International Group for the Psychology of Mathematics Education*, Umeå(Sweden), vol. 3, pp. 131–138. PME (2018)
22. Ingarozza, F., Adamo, M.T., Martino, M., Piscitelli, A.: A grossone-based numerical model for computations with infinity: a case study in an Italian high school. In: Sergeyev, Y.D., Kvasov, D.E. (eds.) 3rd Int. Conf. "NUMTA 2019 - Numerical Computations: Theory and Algorithms", LNCS 11973, pp. 451–462. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-39081-5_39
23. Keisler, H.J.: Elementary Calculus: An Approach Using Innitesimals, third ed. Dover Publications (2012).

24. Khan, S.: New pedagogies on teaching science with computer simulations. J Sci Educ Technol **20**, 215–232 (2011)
25. Klymchuk, S., Zverkova, T., Gruenwald, N., Sauerbier, G.: University students' difficulties in solving application problems in calculus: Student perspectives. Mathematics Education Research Journal **22**(2), 81–91 (2010)
26. Knuth, D.E.: Mathematics and Computer Science: Coping with Finiteness. Science **194**(4271), 1235–1242 (1976). https://doi.org/10.1126/science.194.4271.1235
27. Lithner, J.: Students' mathematical reasoning in university textbook exercises. Educational Studies in Mathematics **52**, 29–55 (2003)
28. Lithner, J.: Mathematical reasoning in calculus textbook exercises. Journal of Mathematical Behavior *23*, 405–427 (2004)
29. Lithner, J.: University Students' Learning Difficulties. Education Inquiry **2**(2), 289–303 (2011)
30. Muzangwa, J., Chifamba, P.: Analysis of Errors and Misconceptions in the Learning of Calculus by Undergraduate Students. Acta Didactica Napocensia **5**(2), (2012)
31. Rizza, D.: Primi passi nell'aritmetica dell'infinito. Preprint (2019)
32. Sergeyev, Y.D.: A new applied approach for executing computations with infinite and infinitesimal quantities. Informatica **19**(4), 567–96 (2008)
33. Sergeyev, Y.D.: Arithmetic of infinity. Edizioni Orizzonti Meridionali, Cosenza (2003)
34. Sergeyev, Y.D.: Evaluating the exact infinitesimal values of area of Sierpinskinski's carpet and volume of Menger's sponge. Chaos Solitons Fractals **42**(5), 3042–6 (2009)
35. Sergeyev, Y.D.: Higher order numerical differentiation on the Infinity Computer. Optimization Letters **5**(4), 575–85 (2011)
36. Sergeyev Y.D.: http://www.theinfinitycomputer.com (2004)
37. Sergeyev, Y.D.: Lagrange Lecture: Methodology of numerical computations with infinities and infinitesimals. Rendiconti del Seminario Matematico Università e Politecnico di Torino **68**(2), 95–113 (2010)
38. Sergeyev, Y.D.: Solving ordinary differential equations by working with infinitesimals numerically on the Infinity Computer. Applied Mathematics and Computation **219**(22), 10668–81 (2013)
39. Sergeyev, Y.D.: Un semplice modo per trattare le grandezze infinite ed infinitesime. Matematica, Cultura e Società: Rivista dell'Unione Matematica Italiana **8**(1), 111–47 (2015) (in Italian)
40. Sergeyev, Y.D.: Using blinking fractals for mathematical modelling of processes of growth in biological systems. Informatica **22**(4), 55976 (2011)
41. Sergeyev, Y.D.: Numerical infinities and infinitesimals: Methodology, applications, and repercussions on two Hilbert problems. EMS Surveys in Mathematical Sciences **4**(2), 219–320 (2017)
42. Sergeyev, Y.D., Mukhametzhanov, M.S., Mazzia, F., Iavernaro, F., Amodio, P.: Numerical methods for solving initial value problems on the Infinity Computer. International Journal of Unconventional Computing **12**(1), 55–66 (2016)
43. Schoenfeld, A.: On mathematics as sense-making: An informal attack on the unfortunate divorce of formal and informal mathematics. In: Voss, J., Perkins, D., Segal, J. (eds), Informal Reasoning and Education, pp. 311–344. Lawrence Erlbaum Associates, Hillsdale (NJ) (1991)
44. Sierpinska, A.: Humanities students and epistemological obstacles related to limits. Educational Studies in Mathematics **18**, 371–397 (1987)
45. Steinberg, R.N.: Computer in teaching science: to simulate or not simulate? Phys Educ Res: Amer J Phys Suppl **68**(7), S37 (2000)

46. Tall, D.: A child thinking about infinity. Journal of Mathematical Behavior **20**, 7–19 (2001)
47. Tall, D.: Mathematical intuition, with special reference to limiting processes. Proceedings of the Fourth International Conference for the Psychology of Mathematics Education, 170–176 (1980)
48. Tall, D.: The transition to advanced mathematical thinking: Functions, limits, infinity and proof. In: Grouws, D., (ed.), Handbook for Research on Mathematics Teaching and Learning, pp. 495–511. Macmillan New York (1992)
49. Tall, D., Vinner, S.: Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics **12**, 151–169 (1981)
50. Williams S.R.: Models of Limit Held by College Calculus Students. Journal for Research in Mathematics Education **22**(3), 219–236 (1991)