Paradoxes of the infinite and ontological dilemmas between ancient philosophy and modern mathematical solutions

Fabio Caldarola[1] [0000−0002−4011−848X], Domenico Cortese[2], Gianfranco d'Atri[1], and Mario Maiolo[3] [0000−0002−6208−1307]

[1] Dep. of Mathematics and Computer Science, Cubo 31/A, Università della Calabria, Arcavacata di Rende 87036 (CS), Italy
caldarola@mat.unical.it, datri@mat.unical.it
[2] Via degli Orti, 12, 89861 Tropea (VV), Italy
domcor87@hotmail.it
[3] Dep. of Environmental and Chemical Engineering (DIATIC), Cubo 42/B, Università della Calabria, Arcavacata di Rende 87036 (CS), Italy
mario.maiolo@unical.it

Abstract. The concept of infinity had, in ancient times, an indistinguishable development between mathematics and philosophy. We could also say that his real birth and development was in Magna Graecia, the ancient South of Italy, and it is surprising that we find, in that time, a notable convergence not only of the mathematical and philosophical point of view, but also of what resembles the first "computational approach" to "infinitely" or very large numbers by Archimedes. On the other hand, since the birth of philosophy in ancient Greece, the concept of infinite has been closely linked with that of contradiction and, more precisely, with the intellectual effort to overcome contradictions present in an account of Totality as fully grounded. The present work illustrates the ontological and epistemological nature of the paradoxes of the infinite, focusing on the theoretical framework of Aristotle, Kant and Hegel, and connecting the epistemological issues about the infinite to concepts such as the continuum in mathematics.

Keywords: Infinite · Pythagorean School · Greek Philosophy · Grossone · Unimaginable numbers.

1 Introduction

"Mathematics is the science of infinity" says the very famous sentence of Hermann Weyl, 1930 (see [55, pag.17]). And a quite surprising fact is that the South of Italy, in particular Calabria and Sicily, played a historic role of the highest importance in the development of the idea of infinity in mathematics and philosophy, disciplines that at the time were often not distinguishable in many respects. One could almost say that the infinite from the mathematical-philosophical point of view was born in Magna Graecia as well as from the computational point of view, in fact, "unimaginable numbers" were born in the Greek colonies of Sicily by the greatest mathematician of antiquity, Archimedes of Syracuse. It is therefore very interesting to investigate how the concept of infinity was born, developed and evolved on the border between mathematics and philosophy and this is what the present paper wants to do, at least in part. A further coincidence is that recent speculations on the idea and use of infinity, in mathematics but not only, once again see Calabria as the protagonist, as we will see later.

From Section 2, we will analyze and explain why the idea of the infinite has always been perceived as a contradiction in relation with the philosophical and scientific necessity of systematizing the entire reality within a complete and univocal set of measures and forces. Such a contradiction took two forms:

1 - The idea of infinite as a being which pushes itself beyond any given limit in size or time. Once one assumes an infinite being, there is no longer a way to hypothesize a "center" in order to univocally identify the position of the parts in the existent, nor there is a way to understand a foundation which explains being in its entirety, since entirety can never be comprehended and, therefore, led back to one picture or principle.

2 - The idea of infinite as a process which pushes itself beyond any given limit in size or time. This is the case of the infinite divisibility of matter or of time. Similarly to the preceding case, there is no way to identify a unity of measure (such as integer numbers) through which to construct the entire reality: there will always be a possible incommensurability.

In the last sections we will see that modern philosophy and mathematics - because of the necessity of supposing both discrete unities of measure and a continuum matter and irrational numbers - cannot overcome this conceptual short circuit which leads to the paradox of the infinite (in which, because of the recalled incommensurability, the infinite processes imply that the "part" is as large as the "whole"). Idealistic philosophy, instead, disarms such a concept of infinity by considering it as an intellectual trap and as an unimportant process from a dialectical or "pragmatic" point of view. The issue becomes to create a convention which puts in harmony all human experiences indifferently to a possible infinite process.

2 The infinite as an ontological problem and the limitless size

The need to resolve *contradictions* in ancient philosophy is the one which is most linked to the typically philosophical necessity of outlining the Totality, that is to say of finding out the unique, original reason or ground (*arche*) which can explain the existence and the behavior of "everything". The occurrence of a contradiction stays in the fact that if something remains unexplained and unconnected to the rest of totality the very notion of knowledge of totality and *episteme* as pursued by philosophy collapses (see, for instance, [14, 17, 52, 56]). This can happen if not everything can be reduced to one principle and we need at least two disconnected ones to make sense of phenomena, or if not all becoming can be explained with one principle. Such an unexplained differences would be arbitrary gaps which would deprive philosophy (or epistemology) of its ultimate goal: to trace the rational origin of reality in order to inscribe all its aspects within "the sense of the Whole". This goal is not at all extraneous to the intrinsic nature of modern science and physics - for instance, in its effort to locate elements which are more and more fundamental and to ultimately unify all physical laws into one great theory (theory of everything, see [21, 23]).

Pythagoras was maybe the first philosopher mathematician who really had to deal with the concept of infinite and with the disruption of the idea of a "structured whole" which its existence entails. Pythagorean mathematics is based on the idea of "discontinuity", as it is exclusively anchored in integer numbers and, therefore, the increase of a magnitude proceeds by "discontinuous leaps". In such a worldview all objects were constituted by a finite number of monads, particles similar to atoms. Two magnitudes could be expressed by an integer number and were mutually commensurable, they admitted a common denominator. Pythagorean thought will be put in crisis by the discovery of incommensurable magnitudes (that is to say, which do not admit a common denominator), developed within the school itself as the relationship between diagonal and side of a square resulted to be irrational, and safeguarded as an unspeakable secret. This entailed that diagonal and side are composed not by a finite amount of points, but by infinite points: for the first time actual infinite and not only potential infinite was discussed.

Within this conceptual framework, the idea of infinite becomes an issue in several senses, which are strongly interrelated. The simplest of these senses is infinite as what pushes itself beyond any given limit in size or time. Can an "infinite" size (or time) be considered consistent with the - metaphysical or even physical - necessity of postulating a Totality which must be, at least in principle, wholly grounded? Is it more consistent to suppose a finite universe, with the danger of a possible call for "something which would always be beyond its Whole", or to envisage an infinite one, with the risk of the impossibility to make sense of a Being which is never complete and, thus, never totally established by one rational foundation? This last is the objection put forward by the Eleatic Parmenides, for whom Being is eternal in the sense of being without past and future, because it cannot come from not-being, but it is hypothesized as a finite sphere - as opposed to the idea of his disciple Melissus. For Parmenides Being is also immutable: becoming can never be rationally justified from the existence of the necessary one original founding ground, because the alteration of this "one" should be explained by another principle otherwise it would be arbitrary and contradictory, and such a double principle should be explained by another original principle and so on. Becoming must only be explained as an illusion. The ultimate paradox of the infinite stays here. If you suppose its existence - in the extension of time and space or in the divisibility of time and space - you **cannot** hypothesize an original element which explains why reality organizes itself in a certain way or in another. In fact, if you suppose an infinity divisibility of time and space you cannot locate or imagine such an original element, and if you suppose an infinite extension of time or space you can never **ensure** the entire consistency of time and space with the structure dictated by such an element. On the other hand, if you suppose the non-existence of the infinite, it would be impossible to explain why a Being stops being, or why a certain force stops acting (in its capacity to divide matter or time). The latter case can only be explained by the existence of two counterposed original elements, but their contraposition would remain unexplained, betraying the very aim of episteme and science.

3 Infinite as infinite divisibility of a certain size or time and the problem of continuum

While Parmenides, maybe the first philosopher to clearly recognize the question above, disarms the paradox of the infinite by excluding - as an "illusion" - the possibility of time, becoming, alteration, infinity but even of something outside the sphere of being (not preventing some logical inconsistency), Aristotle philosophy tries to circumvents the paradox by accepting the duplicity of the original element. In his case, they are "form" and "matter". The "ground" which for the Stagirite is at the basis of ontology, in fact, does not concern so much the mere physical and material constituents of being as, rather, the formal principles which imprint the underline indefinite matter and which build individuals and characters of reality. First of all, in the Aristotelian ontology this formal principle can stand "on its own" in the sense that a form, an identity of a primary substance is complete and it does not draw its characters or existence from other structures. Also, primary substances, which correspond to the individuals, are not mere aggregates of their parts. To stand on its own, in this case, does not mean that substance cannot change. Substance can be generated, altered and corrupted by means of a "substantial transformation" (that is to say through the process in which an individual assumes or loses its form on or from its matter - which is, in turn, an union of another form and another matter until the indefinite primary matter in reached).

The transformations of substances are explained by the four interpretative causes, material, efficient, formal and final one, with the final cause being the crucially prevalent one in explaining the existence and change of things, as clarified in *Physics II* and in *Parts of Animals I.* The causes, in fact, are ultimately explained as an irresistible tension which the worldly matter has towards the supreme immobile mover, a *de facto* divinity which has, for Aristotle, the same qualities of the Eleatic Being and act as final cause of everything. Aristotle, like Plato before him, solves the paradox of movement by assuming a necessary "axiomatic" duplicity of first principle, in his case the supreme first mover and indeterminate matter which continuously needs to be "informed"; a duplicity which explains the relentless finalistic tension which corresponds to becoming. Indeed, Aristotle's metaphysics presents the interesting characteristic of interpreting all kinds of movements, included division and enlargement, as expression of a finalistic push which is itself the tension between the two "fundamental *archaí*" of Being: in this way the contradiction of the existence of movement, division or alteration which the necessity of a unique ground brings is sterilized. Further, one of these two "principles" seems to be, for the Stagirite, only a principle in a negative sense, interpretable as an absence of a definite form (see Physics, 204a8–204a16, and Metaphysics, IX.6, 1048a–b, in [4, Vol. I]).

Such a theoretical scenario entails one important thing: Aristotle cannot be interested in infinite division of a magnitude (and in the infinite backward reduction it implies) as an action corresponding *per se* to an attempt to locate the Origin. The prospect of an infinite reduction is theoretically unimportant because the transformations in which its steps consist (as every kind of transformation) would already be a reflection and a function of what is for him the arche, the tension towards immobile mover. The ontological gaps (in the sense explained in the previous section) which a similar series of actions manifests are already anyway "endless"; in the sense that the inevitable existence of matter - the "second principle" - makes the tension towards perfection endless. What really counts from a scientific and an ontological point of view is, therefore, the comprehension of the nature - that is to say of the causes - of the daily phenomena and their ultimate direction. The existence of infinite division is therefore ontologically relevant in the possible daily progression of its stages (infinite in potentiality), not in its concept as "already actually present" (infinite in actuality), which stops being idealized.

Such a rational solution is no longer possible in the philosophical era which Severino characterizes for the dissolution of the cohesion of "certainty" and "truth" (see [52, vol. II]). Modern philosophy, from Descartes to Kant included, introduced the opposition between "certainty" and "truth": the majority of philosophers had to deal with the fact that what appears rational to human thought may be at odds with what is outside it. Kant fulfills this attitude by claim in that the chain of conditions, and also the world, are a subjective representations and their epistemological value stays in how we manage - by means of sensibility and understanding, the subjective categories - to build a consistent picture of experience. Reason is still an arbiter of empirical truths, useful to assess consistency and harmony of the set of concepts which our understanding produces. But outside experience it does not make sense to rationally argue for one solution or the other of the aporia earlier recalled, for finitude or for infinity of original conditions and constituents: it is outside human reaching, for instance, to verify whether a series of causes is actually infinite, such a totality is never to be met with in experience, it's only the "natural attitude" of human reason to relentlessly chase this task (see [28, pag. 460-461]). As a consequence, the first two antinomies of pure reason involve the very issue of infinity in relation to entirety: *"Thesis 1: The world has a beginning in time, and is also limited as regards space. Antithesis 1: The world has no beginning, and no limits in space; it is infinite as regards both time and space. Thesis 2: Every composite substance in the world is made up of simple parts, and nothing anywhere exists save the simple or what is composed of the simple. Antithesis 2: No composite thing in the* *world is made up of simple parts, and there nowhere exists in the world anything simple."* (See [36, pag. 48] and also [5, 6].)

This coincides to the fact that Kant's philosophy cannot outflank the paradoxes of the infinite or, better to say, he cannot propose a theory of knowledge which metaphysically or even existentially systematizes the world. Once the faculties of reason are limited in their scrutiny of radical backward causes and ontological elements by the inevitable gap between certainty and truth, one cannot claim any definitive solution to potential contradictions. Theoretical reason and categories cannot embrace the knowledge of the first cause and of the totality - and, consequently, of the solutions to the paradoxes of the infinite.

Modern mathematics and modern philosophy seem to share the impossibility to rationally unarm the tension caused by the existence of something which is incommensurable to the notion of entireness and complete systematicity. As hinted before, if one supposes an infinity divisibility of time and space one cannot locate or imagine an original element which structures the fabric of reality with its regularities, and if one supposes an infinite extension of time or space it can never been ensured the entire consistency of time and space with the structure dictated by such an element. In mathematics such a problem corresponds with that of the continuum. If empirical reality is rationally perceived as a "continuum" both in its temporal or spatial extension and in its subdivision, in other words, you can always pick and isolate a further fragment of magnitude which disrupts what has been systematized so far in term of relations among different magnitudes to picture the structure of reality, without rationality being legitimate in restricting such a process with a justifications which has not empirical bases. All this makes reciprocal commensurability among unities of measures and imagines of completeness impossible. Hence the *paradox of the infinite* whereby the part is as "large" as the entire. Mathematics and modern philosophy issues may be synthesized in this way. The character of the incommensurable to entireness and complete systematicity is brought, in mathematics, by the very necessity of systematicity to compare and measure reciprocally incommensurable magnitudes and, in modern philosophy, by the necessity not to resort to ideal rational hypotheses in the origin of Being to contain the logical infinite ontological processes explained in this article.

Because of the existence of the continuum in mathematics and physics, you cannot get rid of the "actual" infinite because you cannot get rid of the incommensurable. Therefore you cannot do without supposing an "already happened infinite approach" which converges to a measure, in order to "perfectly" measure things - as opposed to Aristotle. Limits, or better series, are a way to explicit that entire, structurally finite magnitude to which the sum of some infinite fragments is convergent.

4 Infinite as a "convention" and the conventional resolutions to the contradiction

Infinite has also been thought as resolution to the intellectual and existential discomfort which derives from the finitude of things within becoming. This position has been adopted, in the history of philosophy, first by pre-Socractic thinkers who did not have yet a clear account of the paradox of the infinite as described above and, secondly, by idealistic philosophers such as Hegel who decided that the only way to circumvent such a paradox is to resort to the acceptation of "conventions" as ontologically valid. In other words, infinite is the character of the totality of being once we have given a *reason* to the chaotic turning up of differences, to the apparent contradictions and all kinds of obstacles which make our experience of the world appear full of uncomfortable discrepancies. A similar rationalization aims at finding the intrinsic harmony of things, so to show the ground, the origin (arche) which accounts for their coming into beings and, therefore, describes the Totality as the very Logic of their opposition. A Logic which, taken "in itself", is not limited to one contingent thing, it does not find arbitrary discrepancies: it is *un*limited, like a circle. The philosopher who manages to be aware of that is able to experience the non-contradiction of existence. The rational unity of all things is infinite, therefore, in the sense that it never presents a particular which is impossible to make sense of, and therefore it cannot contemplate a feeling of limit or restriction due to a contradiction: it is always already "full of Being". Such a conception of infinite and arche first appears in Greek philosophy in Anaximander and his idea of apeiron (see [53]), the original identity of all things which is without peculiar determinations and, thus, "without boundaries". It was famously elaborated by Heraclitus and his emphasis on the strive of oppositions as the logic principle of reality (see [27, 29]) and it will be, as we will see, a crucial feature of Hegel's idealism.

German idealism - and its major representative, Hegel - introduces an innovative insight to modern philosophy: "construction" is real, what seems to be the fruit of human mind or human culture is the expression, the reflection or, even, an active part of the rational structure of reality and of its logic. From this point of view it does not make any sense to distinguish between "empirical" and "metaphysical" construction as everything draws its "epistemo-logical dignity" from its being a logical or "practical" passage which contributes to harmonize and rationalize experience. With Hegel we meet the conceptions of the infinite as "rationalization" which aims at finding the intrinsic harmony of things, so to show the origin which accounts for their coming into beings and, therefore, describes the Totality as the very Logic of their opposition. Being infinite, in this sense, is the character of the idealistic Absolute and its ontological sense is the very opposite of the infinite as a limitless addition of *finite* aspects of reality. To continue adding an element to a concept or to a magnitude without elaborating a rational synthesis of the sense of its coming into being or of its *telos* - and the other parts of reality means to maintain a certain "arbitrary" discrepancy. The infinite series of this concept is what Hegel calls "bad infinity" (see [24]).

The contradiction between the necessity to find a first condition to make our conception of the world "complete" on the one hand and the necessity not to "arbitrarily stop" without reason to a certain element (proper of a world which is continuum) on the other hand should be synthesized by the idea of the universe as eternal (without time) and "circular" as rationally explained in itself, as the existence of a feature is always explained as the synthesis of a previous apparent discrepancy, without the need to resort to a condition which is out of the already "existent" dialectic of beings. If a thought manages to invent an instrument which makes sense of a process, harmonize its meaning with the surroundings without resorting to virtually infinite actions, such a thought is real and rational. Even if the cosmological problem is not central to Hegel's reflection, his dialectics brings to the non-necessity of an infinite research of spatial and temporal limits or conditions. As he says in the Science of Logic, *"the image of true infinity, bent back onto itself, becomes the circle"* (see [24, pag. 149, §302]).

The most original and pregnant insight of Hegel's rationalization of reality is that it does not matter, for dialectical speculation, how many "finite" and irrational characters are present in a certain context, or how deep a contradiction is structured in a concept. Since the aim of philosophy - and, therefore, of human existence - is to make thought and Being coincide (see [15]) and since thought and Being are not two separate entities, any practical convention which fulfills the task of making such irrational characters unimportant and of making our acting and experience "satisfactory" has the dignity of a real character of reality. One way to "synthesize" the necessity of modern mathematics calculus of resorting to the concept of incommensurability between magnitudes - and, thus, of limits and "actual" infinite - and the necessity of logic and common sense to avoid the relative paradoxes is the concept of grossone, proposed by Sergeyev. The grossone appears to be what has just be named as a conventional systematization which interprets the infinite as an Entire in order to outflank its contradictions, and which consciously overlooks the fact of its being constituted by "infinite finitudes" because practically unimportant - and for this reason "irrational" in a dialectical sense - in the context of its applications.

Sergeyev starts from a literary interpretation of the principle "the part is less than the whole" and he applies it both to finite and, especially, infinite quantities, in strong contrast with the conception of Cantor's infinite, for which the infinite processes imply that the "part" is as large as the "whole". As we have seen, Idealistic philosophy too, agrees with such a principle that leads to a "multiple concept of infinity", that results also in accord with the dialectical or "pragmatic" point of view. At the University of Calabria, in the early 2000, Y. Sergeyev began to develop his idea of an infinite with good computational properties and, at the same time, easy to use at every level, from schools to academic research. In a few words, Sergeyev developed a new computational system based on two fundamental units or "atoms": the ordinary unit 1 to generate finite quantities as the familiar natural or integer numbers, and a new infinite unit, named *"grossone"*, to generate infinite and infinitesimal numbers. For example, [40, 42, 45, 48, 50] are introductory surveys to this new system and the book [41] is also written in a popular way and well understandable for high school students. This new computational methodology has even been applied in a number of areas both in mathematics, applied sciences, philosophy, and recently also in mathematical education. For instance, optimization and numerical differentiation ([1, 16, 20, 35, 44, 46, 51]), calculus of probabilities and geometry ([13, 33, 34]), fractals and complex systems ([3, 8, 9, 12, 18, 19, 43, 47, 49]), supertasks ([39]), didactic experimentations ([2, 26, 38]), and others.

The peculiar main characteristic of the grossone, that leads it to a growing diffusion, is probably his double nature of infinite number, but with the behavior of an ordinary natural one. This intrinsic characteristic is in fact the basis of several of its properties among which we recall, since this article deals with paradoxes from different points of view, the ability to solve some of them focused on the infinite. For example, consider what is probably the most famous of them, the Hilbert's paradox of the Grand Hotel, which was designed to highlight the strange properties of the infinite countable cardinality $\aleph_0$ in the Cantorian sense. Even when the hotel is full, it is always possible to host a finite or infinite countable number of new guests by moving the present guests in the hotel in a suitable manner.[4] This leads to many semantic problems at various levels, as well as questions of a logical nature. To take a simple example, it is sufficient just to consider the semantics of the word "full": commonly, if a tank or container is full it is not possible to add more. Adopting the grossone system instead, such kinds of paradoxes are avoided because a hotel with ① rooms cannot accept other guests when it is full (see for instance [41, Section 3.4]).

Another example, of different nature, that yields a paradox is the following, known as the *Thompson lamp paradox*.

Example 1 (The Thompson lamp paradox). Given a lamp, assume that we turn it on at time zero and turn it off after 1/2 minute, then we turn it on after 1/4 minute and turn it back off again after 1/8 minute, and so on, by acting on the switch to each successive power of 1/2. At the end of a minute, will the lamp be on or off? This puzzle was originally proposed by the philosopher J.F. Thompson in 1954 to analyze the possibility of completing a supertask, i.e. a larger task made up of an infinite number of simple tasks (see [54]).

Note that the classical geometric series of common ratio 1/2 and starting from 1/2 converges to 1, in symbols

$$\sum_{t=1}^{+\infty} \left(\frac{1}{2}\right)^t = 1, \tag{1}$$

hence in a minute the switch will be moved infinitely many times (admitting that this is possible) and the question above has no answer. Using the grossone-based system, it is offered in [41, Section 3.4] the solution that the lamp is "off" after one minute, and it is essentially due to the parity of grossone.

[4] If there are n new guests the simplest choice is to use the function $\mathbb{N} \to \mathbb{N},\ m \mapsto m+n$, instead, in case of an infinite countable number, the function $\mathbb{N} \to \mathbb{N},\ m \mapsto 2m$ (see [22, 41]).

We propose here a more detailed interpretation: the switch will have the first motion at the time zero, the second at the time $\frac{1}{2}$, the third at the time $\frac{1}{2}+\left(\frac{1}{2}\right)^2$, and the n-th at the time $\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^{n-1}$. We recall that a sequential process, in Sergeyev's theory, cannot have more than ① steps, hence the last action on the switch, i.e. the ①-th action, will be made at the time

$$\sum_{t=1}^{①-1}\left(\frac{1}{2}\right)^t = -1+\sum_{t=0}^{①-1}\left(\frac{1}{2}\right)^t = -1+\frac{1-\left(\frac{1}{2}\right)^{①}}{\frac{1}{2}} = 1-\left(\frac{1}{2}\right)^{①-1}$$

(cf. Eq. (1)). In other words this means that the last action on the switch will be made at an infinitesimal time before the end of the minute, more precisely $\left(\frac{1}{2}\right)^{①-1}$ minute before, and from this moment on, the switch will remain off (hence it will be off as well at the time 1 minute).

Sergeyev, by means of an "infinite" which behaves as an ordinary natural finite number, disarms the anarchy of the continuum. Such an "anarchy" springs from the absence of a limit in a process of division of space or time - or in the growth of a size, absence which is the reason why you cannot locate an original element which can justify a univocal system of references capable of ultimately describing the features of reality or which is the same thing of "creating a stable totality". This infinite process of addition of "fragments of finitude" is what create the inconvenience whereby the "part is as big as the entire" and the consequent impossibility to systematize the entire. To sterilize such a process means to conventionally set aside the traditional and problematic sense of the infinite to embrace what idealistic philosophy and Hegel refers to as "true infinite" (see [24]), that is to say a practical convention which makes thought and experience more fluid and comforting (without ontological "finitudes"). "Infinite" is here used, of course, in a completely unusual acceptation if compared to its common mathematical meaning. The function of grossone is therefore the same as a dialectical (in a philosophical sense) device which ensures that our approach to a certain concept is deprived of discrepancies: we perceive it without ontological "limits" in the sense that we do not find compromising alteration or obstacle in its logic.

Unimaginable numbers[5] seem to respond to the same pragmatic and, therefore, ontological necessity: even if they does not contemplate actually "infinite" quantities, they give "form" to numbers which are as large as to question their

[5] The *unimaginable numbers* are numbers extremely large so that they cannot be written through the common scientific notation (also using towers of exponents) and are behind every power of imagination. To write them some special notations have been developed, the most known of them is *Knuth's up-arrow notation* (see [30]). A brief introduction to these numbers can be found in [10], while more information is contained in [7, 11, 25].

commensurability to a rational systematization of our picture of reality, generating the same discomfort of "relentless process" as an infinite process does. But from a closer point of view, these notions can assume a more specific ontological connotation, which communicate with Aristotle's approach as we have already hinted.

We have seen how Aristotle is not interested in infinite division of a magnitude (and in the infinite backward reduction it implies) as an action corresponding *per se* to an attempt to locate the Origin. The prospect of an infinite reduction is theoretically unimportant for him because the transformations in which its steps consist (as every kind of transformation for Aristotle's philosophy) would already be a function of what is the arche, the tension towards immobile mover, the *de facto* divinity external to the empirical perimeter, at the origin of al causes.

Aristotle's philosophy can afford to outflank the issues about the infinite because to conceive it would be impossible and redundant since this very idea is incommensurable to the idea of first cause and first principle, whose action will always be "in progress". The infinite is always only potential, never actual.

The rejection of an ontological hypostatization of the concept of the infinite is also confirmed in its relation to the notion of substance. The infinite is not a formal principle which "is" in the primary sense of the term, it is not a substance whose structure cannot be divided because if so it would lose its unique and original identity: it is an accident which can be "divided" and may potentially happen and be altered by applying to completely different circumstances: *"it is impossible that the infinite should be a thing which is in itself infinite, separable from sensible objects. If the infinite is neither a magnitude nor an aggregate, but is itself a substance and not an accident, it will be indivisible; for the divisible must be either a magnitude or an aggregate. But if indivisible, then not infinite, except in the way in which the voice is invisible. But this is not the way in which it is used by those who say that the infinite exists, nor that in which we are investigating it, namely as that which cannot be gone through."*[6] The infinite is reduced to what we would call today "an idea in a Kantian sense" - with the difference that, as said, its paradoxes does not even have a crucial role in the determination of ultimate metaphysical truths: *"the infinite does not exist potentially in the sense that it will ever actually have separate existence; it exists potentially only for knowledge. For the fact that the process of dividing never comes to an end ensures that this activity exists potentially, but not that the infinite exists separately."*[7] Similarly, the unimaginable numbers can be interpreted as an idea in "a Kantian sense", having only a regulatory function in the sense that they own a linguistic and rational denotation and can give a sensible, "reasonable" context to the regularities which we ordinarily experiences, but that will never actually exist because no thoughts or no computing machine will never be able to "see" (or even think) them. Obviously, in the logical context of modern mathematics, the idea of original immobile mover is note relevant as the

[6] See Physics, 204a8-204a16, in [4, Vol. I].
[7] See Metaphysics, IX.6, 1048a-b, in [4, Vol. I].

concept of "natural number", which share with the first one a similar regulatory theoretical (and, therefore, pragmatic) "action".

5 Conclusions

The sense of the introduction of grossone into a mathematical formula cannot be reduce to a "technical stratagem", and to understand it it is necessary to understand the ontological relevance of the problem of the infinite in the context of a consistent picture of the world which any discipline try to achieve. In this paper we have seen how the original philosophical effort to comprehend the entireness of reality, both in the sense of its fundamental elements which justify its structure and in the sense of the "total amount" of its features, inevitably clashes with the problem that the process of locating these elements or this totality can be thought as without an end. This problem produces the paradox whereby a "part" of reality can contain as many amounts of hypothetical "fundamental elements" or features as the entire. This is an issue which belongs to ontology in a broad sense as to mathematics and its elements in a more specific acceptation, since mathematics - in order to work and to be applied to "experience" - has to create its specific reality consistent in its entireness and in its axioms. One of the solutions to this philosophical paradox elaborated by modern philosophy is to take this problem as a very "problem of thought", making reality coincide with our very conception of reality and with our practical and cultural instruments. In this sense, for Hegel's philosophy, reality is dialectical not "for our thought": reality coincides with our thought and *vice versa*, and any ontological problem is reduced to a problem of internal consistency and contradiction. But if thought and reality are two sides of the same coin, then to create a logic which allows our thought to work with the "reality" of a discipline without any logical arrest or short circuit which would generate practical issues is to create a new dialectical advance in reality - or, at least, in the reality internal to our discipline. The innovation of grossone is that it seems to engage with such a task in the context of the paradoxes of the infinite, by proposing an explicitly "conventional" notion which challenges the very necessity of the ontological mathematical turmoil experienced so far.

Aknowledgments

This work is partially supported by the research projects *"IoT&B, Internet of Things and Blockchain"*, CUP J48C17000230006, POR Calabria FESR-FSE 2014-2020.

References

1. Amodio, P., Iavernaro, F., Mazzia, F., Mukhametzhanov, M.S., Sergeyev, Y.D.: A generalized Taylor method of order three for the solution of initial value problems

in standard and infinity floating-point arithmetic. Mathematics and Computers in Simulation **141**, 24–39 (2017)
2. Antoniotti, A., Caldarola, F., d'Atri, G., Pellegrini, M.: New approaches to basic calculus: an experimentation via numerical computation. In: Sergeyev, Y.D., Kvasov, D.E. (eds.) 3rd Int. Conf. "NUMTA 2019 - Numerical Computations: Theory and Algorithms", LNCS, vol. 11973, pp. 329–342. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-39081-5_29
3. Antoniotti, L., Caldarola, F., Maiolo, M.: Infinite numerical computing applied to Hilbert's, Peano's, and Moore's curves. To appear in Mediterranean Journal of Mathematics.
4. Barnes, J. (ed.): The Complete Works of Aristotle. The Revised Oxford Translation. Vol. I and II, 4th printing, Princeton University Press, Princeton, N.J. (1991)
5. Bennet, J.: Kant's Dialectic. Cambridge University Press, Cambridge (1974)
6. Bird, G.: Kant's Theory of Knowledge: An Outline of One Central Argument in the Critique of Pure Reason. Routledge & Kegan Paul, London (1962)
7. Blakley, G.R., Borosh, I.: Knuth's iterated powers. Advances in Mathematics **34**(2), 109–136 (1979). https://doi.org/10.1016/0001-8708(79)90052-5
8. Caldarola, F.: The Sierpiński curve viewed by numerical computations with infinities and infinitesimals. Applied Mathematics and Computation **318**, 321–328 (2018). https://doi.org/10.1016/j.amc.2017.06.024
9. Caldarola, F.: The exact measures of the Sierpiński d-dimensional tetrahedron in connection with a Diophantine nonlinear system. Communications in Nonlinear Science and Numerical Simulation **63**, 228–238 (2018). https://doi.org/10.10.16/j.cnsns.2018.02.026
10. Caldarola, F., d'Atri, G., Maiolo, M.: What are the unimaginable numbers? Submitted for publication.
11. Caldarola, F., d'Atri, G., Mercuri, P., Talamanca, V.: On the arithmetic of Knuth's powers and some computational results about their density. In: Sergeyev, Y.D., Kvasov, D.E. (eds.) 3rd Int. Conf. "NUMTA 2019 - Numerical Computations: Theory and Algorithms", LNCS, vol. 11973, pp. 381–388. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-39081-5_33
12. Caldarola, F., Maiolo, M., Solferino, V.: A new approach to the Z-transform through infinite computation. Communications in Nonlinear Science and Numerical Simulation **82**, 105019 (2020). https://doi.org/10.1016/j.cnsns.2019.105019
13. Calude, C.S., Dumitrescu, M.: Infinitesimal Probabilities Based onGrossone. Spinger Nature Computer Science 1:36 (2020). https://doi.org/10.1007/s42979-019-0042-8
14. Cavini, W.: Ancient Epistemology Naturalized. In Gerson, L.P. (ed.) Ancient Epistemology. Cambridge University Press, Cambridge (2009)
15. Cesa, C.: Guida a Hegel. Laterza, Bari (1997)
16. Cococcioni, M., Pappalardo, M., Sergeyev, Y.D.: Lexicographic Multi-Objective Linear Programming using Grossone Methodology: Theory and Algorithm. Applied Mathematics and Computation **318**, 298–311 (2018)
17. Crivelli, P.: Aristotle on Truth. Cambridge University Press, Cambridge (UK) (2004)
18. D'Alotto, L.: A classification of one-dimensional cellular automata using infinite computations. Applied Mathematics and Computation **255**, 15–24 (2015)
19. D'Alotto, L.: Cellular automata using infinite computations. Applied Mathematics and Computation **218**(16), 8077–82 (2012)
20. De Leone, R.: The use of Grossone in Mathematical Programming and Operations Research. Applied Mathematics and Computation **218**(16), 8029–38 (2012)

21. David Peat, F.: Superstrings and the Search for the Theory of Everything. Contemporary Books, Chicago (1988)
22. Faticoni, T.G.: The mathematics of infinity: A guide to great ideas. John Wiley & Sons, Hoboken NJ (2006)
23. Gribbin, J.: The Search for Superstrings, Symmetry, and the Theory of Everything. Back Bay Books, New York (2000)
24. Hegel, G.W.F.: The Science of Logic. Di Giovanni, G. (ed.), Cambridge University Press, Cambridge (2010 [1817])
25. Hooshmand, M.H.: Ultra power and ultra exponential functions. Integral Transforms and Special Functions **17**(8), 549–558 (2006). https://doi.org/10.1080/10652460500422247
26. Ingarozza, F., Adamo, M.T., Martino, M., Piscitelli, A.: A grossone-based numerical model for computations with infinity: a case study in an Italian high school. In: Sergeyev, Y.D., Kvasov, D.E. (eds.) 3rd Int. Conf. "NUMTA 2019 - Numerical Computations: Theory and Algorithms", LNCS, vol. 11973, pp. 451–462. Springer, Cham (2020). https://doi.org/10.1007/978-3-030-39081-5_39
27. Kahn, C.H.: The art and thought of Heraclitus. Cambridge University Press, Cambridge (UK) (1979)
28. Kant, E.: Critique of pure reason, transl. Guyer, P., Wood, A.W. (eds.) Cambridge University Press, Cambridge (UK) (1998 [1791])
29. Kirk, G.S.: Heraclitus: The Cosmic Fragments. Cambridge University Press, Cambridge (UK) (1978)
30. Knuth, D.E.: Mathematics and Computer Science: Coping with Finiteness. Science **194**(4271), 1235–1242 (1976). https://doi.org/10.1126/science.194.4271.1235
31. Lolli, G.: Infinitesimals and infinites in the History of Mathematics: a Brief Survey. Applied Mathematics and Computation **218**(16), 7977–8216 (2012). https://doi.org/10.1016/j.amc.2011.08.092
32. Lolli, G.: Metamathematical investigations on the theory of grossone. Applied Mathematics and Computation **255**, 3–14 (2015)
33. Margenstern, M.: An application of Grossone to the study of a family of tilings of the hyperbolic plane. Applied Mathematics and Computation **218**(16), 8005–18 (2012)
34. Margenstern, M.: Fibonacci words, hyperbolic tilings and grossone. Communications in Nonlinear Science and Numerical Simulation **21**(1-3), 3–11 (2015)
35. Mazzia, F., Sergeyev, Y.D., Iavernaro, F., Amodio, P., Mukhametzhanov, M.S.: Numerical methods for solving ODEs on the Infinity Compute. In: Sergeyev, Y.D., Kvasov, D.E., DellAccio, F., Mukhametzhanov, M.S. (eds.) 2nd Int. Conf. "NUMTA 2016 - Numerical Computations: Theory and Algorithms", AIP Conf. Proc., vol. 1776, 090033. AIP Publ., New York (2016). https://doi.org/10.1063/1.4965397
36. Nicolau, M.F.A., Filho, J.E.L.: The Hegelian critique of Kantian antinomies: an analysis based on the Wissenchaft der Logik. International Journal of Philosophy **1**(3), 47–50 (2013)
37. Rizza, D.: A study of mathematical determination through Bertrand's Paradox. Philosophia Mathematica **26**(3), 375–395 (2018)
38. Rizza, D.: Primi passi nell'aritmetica dell'infinito. Preprint (2019)
39. Rizza, D.: Supertasks and numeral system. In: Sergeyev, Y.D., Kvasov, D.E., Dell'Accio, F., Mukhametzhanov, M.S. (eds.) 2nd Int. Conf. "NUMTA 2016 - Numerical Computations: Theory and Algorithms", AIP Conf. Proc., vol. 1776, 090005. AIP Publ., New York (2016). https://doi.org/10.1063/1.4965369
40. Sergeyev, Y.D.: A new applied approach for executing computations with infinite and infinitesimal quantities. Informatica **19**(4), 567–96 (2008)

41. Sergeyev, Y.D.: Arithmetic of infinity. Edizioni Orizzonti Meridionali, Cosenza (2003)
42. Sergeyev, Y.D.: Computations with grossone-based infinities. In: Calude, C.S., Dinneen, M.J. (eds.) Unconventional Computation and Natural Computation: Proc. of the 14th International Conference UCNC 2015. LNCS, vol. 9252 pp. 89–106. Springer, New York (2015)
43. Sergeyev, Y.D.: Evaluating the exact infinitesimal values of area of Sierpinskinskis carpet and volume of Mengers sponge. Chaos Solitons Fractals **42**(5), 3042–6 (2009)
44. Sergeyev, Y.D.: Higher order numerical differentiation on the Infinity Computer. Optimization Letters **5**(4), 575–85 (2011)
45. Sergeyev, Y.D.: Lagrange Lecture: Methodology of numerical computations with infinities and infinitesimals. Rend Semin Matematico Univ Polit Torino **68**(2), 95–113 (2010)
46. Sergeyev, Y.D.: Solving ordinary differential equations by working with infinitesimals numerically on the infinity Computer. Applied Mathematics and Computation **219**(22), 10668–81 (2013)
47. Sergeyev, Y.D.: The exact (up to infinitesimals) infinite perimeter of the Koch snowflake and its finite area. Communications in Nonlinear Science and Numerical Simulation **31**, 21–29 (2016)
48. Sergeyev, Y.D.: Un semplice modo per trattare le grandezze infinite ed infinitesime. Mat Soc Cultura: Riv Unione Mat Ital **8**(1), 111–47 (2015)
49. Sergeyev, Y.D.: Using blinking fractals for mathematical modelling of processes of growth in biological systems. Informatica **22**(4), 55976 (2011)
50. Sergeyev, Y.D.: Numerical infinities and infinitesimals: Methodology, applications, and repercussions on two Hilbert problems. EMS Surveys in Mathematical Sciences **4**(2), 219–320 (2017)
51. Sergeyev, Y.D., Mukhametzhanov, M.S., Mazzia, F., Iavernaro, F., Amodio, P.: Numerical methods for solving initial value problems on the Infinity Computer. International Journal of Unconventional Computing **12**(1), 55–66 (2016)
52. Severino, E.: La Filosofia dai Greci al nostro Tempo, vol. I, II, III. RCS Libri, Milano (2004)
53. Theodossiou, E., Mantarakis, P., Dimitrijevic, M.S., Manimanis, V.N., Danezis, E.: From the Infinity (Apeiron) of Anaximander in ancient Greece to the Theory of infinite Universes in modern Cosmology. Astronomical and Astrophysical Transactions **27**(1), 162–176 (2011)
54. Thompson, J.F.: Tasks and super-tasks. Analysis **15**(1), 1–13 (1954)
55. Weyl, H.: Levels of Infinity/Selected Writings on Mathematics and Philosophy, Peter Pesic (ed.). Dover (2012)
56. Zanatta, M.: Profilo Storico della Filosofia Antica, Rubettino, Catanzaro (1997)