A new approach to the Z-transform through infinite computation

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Abstract

The Z-transform is an important mathematical tool to model sample-data control systems or other discrete-data systems. Since the Z-transform is defined as sum of an infinite number of addends, it is very interesting to look at it from non-classical points of view through one of the many current theories that today provide a wide range of different infinities and infinitesimals. In this paper, therefore, we choose to adopt a new simple applied approach recently proposed by Y.D. Sergeyev that allows one to execute easily numerical computations with various sizes of infinite and infinitesimal numbers. Using this new approach, we obtain a very different type of Z-transform of a complex sequence (or better, a family of infinitely many Z-transforms attached to the same sequence) whose existence is guaranteed almost everywhere on $\mathbb{C}$, unlike what happens in traditional analysis in which the bilateral Z-transform often does not exist anywhere.

Keywords: Z-transform, Formal power series, Region of convergence, Infinities and infinitesimals, Grossone, Supercomputing.

1. Introduction

In mathematics, physics, engineering and other fields of applied sciences, various types of transforms such as integral transforms, Laplace transform, star transform, Fourier transform and discrete Fourier transform, Zak transform etc. are used, depending on the problem-whether discrete or continuous case. Discrete systems cannot be studied using the classical Laplace or Fourier transform because they require continuous (or at least integrable) functions; such systems, instead, can easily be modeled using the Z-transform (see for example [2, 14, 17, 18, 23, 36, 39, 40, 41]). The idea of the Z-transform was first known to Laplace and was later introduced by W. Hurewicz as a controllable way of solving linear, constant-coefficient difference equations (see [24, 41]). In mathematical literature, the idea contained in the Z-transform is also referred to as the method of generating functions as introduced by de Moivre with regards to probability theory (see [23]).

As regards the applications of the Z-transform, the most known ones are in signal processing, digital control and geophysics, but actually it is very widely used in all fields of applied sciences and engineering. The authors are particularly interested in applications in hydraulic engineering to handle discrete data systems arising from water consumptions and water requests in real time within water networks, and several works and problems in this direction are in progress or under investigation. The recent study [42] instead, employs the Z-transform in relation to surface runoff for storm event data.

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From a strictly mathematical point of view, the Z-transform is simply a power series representation of a discrete-time sequence, and therefore it exists if the series converges. But since it is defined as sum of an infinite number of items, it becomes very interesting to consider the Z-transform in relation to atypical numerical systems and non-standard theories that admit different types of countable (and uncountable) infinities and, in recent years, are experiencing a new impetus. We want to immediately clarify that this paper should be understood as a (first?) experimental work on the subject, with primarily computational purposes. Moreover, as far as it will be dealt with here, it is not essential to use a particular numbers system rather than another; instead, what is really important is, in substance, that it is a numerical framework that distinguishes as distinct infinities, contents of the type

$$\text{infinity}, \quad \text{infinity} + 1, \quad -\text{infinity} - 4, \quad 2 \cdot \text{infinity}, \quad \text{infinity}^2, \quad \text{infinity}^5 - 3\,\text{infinity}^2 + 4, \quad \text{etc.} \tag{1}$$

Among the many theories existing today that admit infinities and infinitesimals, probably the most known is *non-standard analysis* developed by A. Robinson in the early 60s using a model-theoretic approach (see [44, 45]). In the sebsequent decades many authors tried to simplify the high technical (expecially from the logical point of view) Robinson's machinery, developing more easy-to-use settings or alternative theories; we recall, for instance, the pioneering notes of W.A.J. Luxemburg (see [30]), the simpler semantic approach, due to E. Zakon and Robinson himself, realized through the introduction of purely set-theoretical objects named *superstructures* (see [46]), the completely different path (syntactic approach) undertaken by E. Nelson in the 70s that led to the formulation of the milestone called *Internal Set Theory* (see [35]), the *elementary axiomatics* by H.J. Keisler (see [25]) whose textbook [26] is currently adopted in many calculus courses in several countries, the approach through the *Alternative Set Theory* as developed by P. Vopěnka in the last 70s and 80s (see [67]), the algebraic laboratory firstly suggested by W.S. Hatcher in [19] and successively pursued and generalized by V. Benci and M. Di Nasso (see [3, 5]), the *constructive* approach by E. Palmgren (see [37, 38]), the "gentle" presentation of C.W. Henson (see [20]), and others.

We reserve a separate mention for the interesting and readable further axiomatic approach proposed by V. Benci and M. Di Nasso called *Alpha-Theory*: without using too technical tools, the authors rigorously describe the addition to the set of natural numbers of an "ideal" element α having the properties of an "infinitely large" number (see [4] for all the details or also [6] for a useful rather informal survey explaining eight different original approaches to nonstandard analysis).

If all the previous architectures (and not only these, think also to the *Levi-Civita field*, to the *superreal* and *surreal numbers*, etc.) share a basic planning that starts from the theoretical logical-deductive formalism and then looks towards applications, a new *numerical* methodology to treat infinite numbers, directly arising from applicative and computational contexts, was introduced in the early 2000s by Y.D. Sergeyev. Drawing inspiration from "the physical world around us"[1] he developed a numerical computational system based on *two fundamental atoms*, the ordinary natural number $1 \in \mathbb{N}$ for finite quantities, and a new infinite unit ① called *grossone* for infinite and infinitesimal quantities: the reader can see [49, 52, 57, 58, 60, 62] for detailed introduction surveys, the book [47] written in a popular way or [29, 34] for more technical insights. This new numerical system has already been applied in many different research areas of pure and applied mathematics, and also of several experimental sciences: for instance, in optimization and numerical differentiation (see [15, 16, 53, 69]), in Euclidean and hyperbolic geometry (see [31, 32]), fractals (see [9, 10, 48, 50, 55, 59, 66]), cellular automata (see [12, 13] and in the context of [7] under investigation), numerical solution of ordinary differential equations (see [1, 33, 56, 65]), infinite series (see [51, 54, 61, 68]), percolation (see [21, 22, 66]), Turing machines and supertasks (see [43, 63, 64]), etc. Instead, as far as we know, in the present paper the new system is used for the first time in connection with complex numbers and variables, and also with doubly-infinite series.

As we will see later, such a new computational methodology turns out fruitful also with respect to the Z-transform and, in particular, our approach allows one to do a more precise and satisfactory analysis on the existence of the Z-transform of a sequence, when we view it in such a new scenario. For instance, we will get that

- given the sequences in Subsections 3.1–3.3, the *bilateral Z-transform* exists for every nonzero value z of the complex plane. This is a very unusual fact in traditional analysis because the bilateral Z-transform often does

[1] It is a frequent expression in the papers of Sergeyev when he speaks of the aims and intentions of his method. It is also interesting to compare with [28, Section 9].

not exist anywhere, and when it exists, is never defined, except very special cases, on such large domains as the whole complex plane, but usually, in an annulus $A = \{z \in \mathbb{C} : r < |z - z_0| < R\}$ for some non-negative $r, R \in \mathbb{R} \cup \{+\infty\}$ with $r \le R$ (see Section 3);

- the Z-transform is no longer unique, but it is replaced by a whole family of infinitely many functions living in the new environment. In fact, many different Z-transform functions, arising for different infinite numbers of the new computational system, can be distinguished starting from the same sequence;
- some classical well known properties of the Z-transform do not require appreciable corrections, while others must be significantly modified to continue to hold (see Section 4).

The paper is structured as follows: in Section 2, after recalling some essential definitions, we present briefly the new methodology for treating sequences, infinite sums (Subsection 2.1) and in particular, for the first time, double infinite sums (Subsection 2.2). Examples 2.2 to 2.6 give explicit computations in some cases.

Section 3 opens the central part of the work and deals with the concept and the theory of Z-transform, by starting from its definition both in the usual context and in the new grossone framework. In the following three subsections we use this numerical system to calculate explicitly the Z-transform in some enlightening cases which are proposed as archetypes and models to follow for general computation. In particular, we consider "the exponential sequence" in Subsection 3.1, "the unit ramp sequence" in Subsection 3.2 and "the parabola sequence" in Subsection 3.3, making a deep analysis on each of them and comparing the results arising in the new setting with the ones obtained in traditional mathematical analysis.

Section 4 investigates validity and changes of some classical properties adapted to the proposed view of the Z-transform, and Section 5, finally, handles the conclusions.

As regards the notations used in this paper, we inform that we will use $\mathbb{N}$ for the set of positive integers and $\mathbb{N}_0$ to include also zero. Moreover, as usual, we denote a sequence by $\{a_n\}_n$ or $\{a_n\}$, or sometimes simply by a_n.

2. A new methodology for dealing with infinity: sequences, sums, and double infinite sums in the grossone framework

From here onwards we will tacitly assume that the reader is familiar with Sergeyev's numerical system, knows how to develop computations and how to use the basic notions and properties. Numerous examples and extensive discussions on the new framework can be easily found in [47, 48, 49, 50, 51, 52, 54, 55, 57, 58, 59, 68] and in the references therein. It is very important to also remark that all the proofs present in the paper are purely computational. This means that a less experienced reader can replace, in a first reading, the symbol ① with an exceptionally large number, written for example in Knuth's notation (see [8] or [27]), and follow the calculations and arguments step by step. Instead, here we just recall some nomenclatures and notations frequently used in the literature mentioned above and remind the reader that in Sergeyev's system ① is the number of elements of $\mathbb{N}$ and $n \le$ ① for all $n \in \mathbb{N}$.

The *extended set of natural numbers*, denoted by $\widehat{\mathbb{N}}$, can be written as

$$\begin{aligned} \widehat{\mathbb{N}} &= \Big\{ \underbrace{1, 2, \ldots, ① - 1, ①}_{\mathbb{N}}, ① + 1, \ldots, 2①, \ldots, 3①, \ldots, \\ &\quad ①^2, ①^2 + 1, \ldots, ①^2 + ①, \ldots, 2①^2, \ldots, ①^3, \ldots, 2^{①}, \ldots, 2^{①} + ①, \ldots\ldots, ①^{①}, \ldots\ldots \Big\}, \end{aligned}$$

the *extended set of integers* as $\widehat{\mathbb{Z}} = \widehat{\mathbb{N}} \cup \{0\} \cup \{-N : N \in \widehat{\mathbb{N}}\}$, and the *extended set of non-negative integers* obviously as $\widehat{\mathbb{N}}_0 = \widehat{\mathbb{N}} \cup \{0\}$. Similarly, $\widehat{\mathbb{R}}$ denotes the so-called *extended set of real numbers*; here we are not interested to discuss formal definitions of such a set (which are also object of current investigations from a foundational point of view), because the observation that any expression (that make sense) involving real numbers and ① is certainly contained in $\widehat{\mathbb{R}}$, suffices for our purposes. Finally, the *extended set of complex numbers* $\widehat{\mathbb{C}}$ can be trivially defined as $\widehat{\mathbb{C}} = \widehat{\mathbb{R}} + i\widehat{\mathbb{R}}$. We inform however the reader that in this paper we will deal (except the last formula of Example 2.1) only with powers of the kind $①^a$ and z^b, where $a \in \mathbb{Z}$ is an ordinary integer, z belongs to $\mathbb{C}$ or $\widehat{\mathbb{C}}$, and $b \in \widehat{\mathbb{Z}}$ is an extended integer.

After such fundamental notations, in this section we want to draw attention to the meaning and to the computational use of sequences, infinite sums and double infinite sums in the grossone world (see Subsections 2.1 and 2.2, respectively).

2.1. Sequences and sums in the ①-framework

We expose now some essential facts about sequences and sums in the grossone framework, beginning with the following remark which is very important to remember, when one starts working in the new context.

Remark 2.1.

(i) If $\{a_n\}_n$ is a sequence of any kind of numbers (real, complex, or belonging to some extended set like $\widehat{\mathbb{R}}$ or $\widehat{\mathbb{C}}$), then it is essential to keep well in mind that, in the grossone theory, **there does not exist** a limit of this sequence in traditional (i.e. Weierstrass) sense, but simply the **evaluation of** a_n **at** $n = ①$, in complete analogy with the evaluation of a_n for $n = 10$ or $n = 128$, for example.

(ii) If $\{a_n\}_n$ is any sequence of numbers (real, complex or expressed in the ①-based system), a general definition for the writing $\sum_{n=1}^{①} a_n$, that is suitable for both theoretical and operational purposes, is not an easy question and such kind of problems are outside the aims of the present article. So, in this paper, we will consider sums of the form $\sum_{n=1}^{①} a_n$, only if there is a clearly understood closed formula that expresses the sequence of the partial sums $\{s_m := a_1 + \ldots + a_m\}_m$. Then, developing the evaluation of the sequence $\{s_m\}_m$, in the obvious way, until the ①-th term, and recalling what we just said in (i), we set by definition

$$\sum_{m=1}^{①} a_m := s_{①}. \tag{2}$$

(iii) We recall that, in the new scenario, a sequence can have no more than ① elements, but since an infinite sum can be divided into a collection of sequential summation processes (or can be summed up in parallel), then it may have a number of summands much higher than ①; for a detailed treatment we refer to [47, 49, 51, 52, 54, 57, 58]. We also notice that all the examples stated in this paper for ① or any infinite positive integer like $3①^2 - 6$, can be easily generalized for all numbers $N \in \widehat{\mathbb{N}}$. The same holds for $-①$ or negative infinite numbers like $-4①^5 + 2①$, which can be effortlessly replaced *mutatis mutandis* by $-M$, where M is any element of $\widehat{\mathbb{N}}$. However, in many cases we will prefer to develop explicit computations with concrete numbers of the new system, rather than using arbitrary parameters $N, M \in \widehat{\mathbb{N}}$ which could hide some aspects that, on the contrary, we want to emphasize.

Now that the meaning of an infinite sum has been conveniently explained, it is clear that a writing as the one appearing in the left hand side of (2) depends strongly on the last and not only on the first summand as typically happens for the series in traditional analysis: the following example deals with the easy case of the geometric series but it is illuminating and explanatory for more complex situations. Furthermore, it will be heavily used in the coming section of the paper, as well as in many areas of mathematics.

Example 2.1. Let us consider the geometric series $\sum_{k=0}^{+\infty} q^k$. Traditional analysis proves that it converges to $\frac{1}{1-q}$ only if $|q| < 1$, while, with the grossone-based system, we are able (see [47, 52, 60]) to give a neat answer for all values of $q \in \widehat{\mathbb{C}}$, $q \neq 1$. In fact, we can evaluate the well known formula

$$\sum_{k=0}^{n} q^k = 1 + q + q^2 + ... + q^n = \frac{1 - q^{n+1}}{1 - q}, \tag{3}$$

for any finite or infinite positive integer n and for any quantity $q \neq 1$, finite or infinite as well. For instance, we can easily compute

- $\sum_{n=0}^{①} \left(-\frac{1}{4}\right)^n = \frac{4}{5} + \frac{1}{5} \cdot \left(\frac{1}{4}\right)^{①};$
- $\sum_{n=1}^{①+3} \left(-\frac{1}{4}\right)^n = -1 + \sum_{n=0}^{①+3} \left(-\frac{1}{4}\right)^n = -\frac{1}{5} - \frac{1}{5} \cdot \left(\frac{1}{4}\right)^{①+3};$
- $\sum_{n=0}^{①/2-4} (① + 1)^n = \frac{(① + 1)^{①/2-3} - 1}{①}.$

Similarly to the previous example, we can compute in the new way many other infinite sums that gave just $+\infty$, $-\infty$, or were indeterminate in classical mathematics.

2.2. Double infinite sums in the new setting

The reader at this point, should have tangibly realized the significance and the behavior of a sum like that in the left hand side of (2); but an expression of the following nature

$$\sum_{n=-①}^{①} a_n \tag{4}$$

is not the same thing of that in (2). In fact (4) poses similar problems of interpretation as the *double-ended series* $\sum_{n=-\infty}^{+\infty} a_n$ (written also $\sum_{n\in\mathbb{Z}} a_n$) does in the classical setting. Thus, first of all we recall a widespread definition, commonly used especially in complex analysis when Laurent series are introduced.

Definition 2.1. Let $\{a_n\}_n$ be a sequence of complex numbers; the double-ended series $\sum_{n=-\infty}^{+\infty} a_n$ *converges to* $\alpha \in \mathbb{C}$ if

(a) $\sum_{n=0}^{+\infty} a_n$ converges to some $\beta \in \mathbb{C}$,

(b) $\sum_{n=1}^{+\infty} a_{-n}$ converges to some $\gamma \in \mathbb{C}$, and

(c) $\beta + \gamma = \alpha$.

From the previous definition, the convenient meaning to confer to (4) should be clear, but, since here it is explicitly treated for the first time, we give a formal definition for future references. Recall that the set of extended complex numbers $\widehat{\mathbb{C}}$ contains naturally the field of ordinary complex numbers.

Definition 2.2. Given $M, N \in \widehat{\mathbb{N}}$ (finite or infinite) and $a_n \in \widehat{\mathbb{C}}$, we say that the sum $\sum_{n=-M}^{N} a_n$ is *equal to* $\alpha \in \widehat{\mathbb{C}}$ if, by definition,

(a) $\sum_{n=0}^{N} a_n$ is equal to some $\beta \in \widehat{\mathbb{C}}$,

(b) $\sum_{n=1}^{M} a_{-n}$ is equal to some $\gamma \in \widehat{\mathbb{C}}$, and

(c) $\beta + \gamma = \alpha$.

It is important to note the differences of language between the previous definitions: in Definition 2.1 we speak of "convergence" of a series as so customary in mathematical analysis, instead, in Definition 2.2 it is expressly said that the sum having a fixed infinite number of addends is **exactly equal** to some number that can be infinite, finite or infinitesimal.

Before to see some numerical examples, we point out the following remark which is trivial also in the new context: this means that, obviously, a sum like that in the left hand side of (4) does not depend from the choice of $n = 0$ to split it.

Remark 2.2. Let $M, N \in \widehat{\mathbb{N}}$, $a_n \in \widehat{\mathbb{C}}$ and $n_0, n_1 \in \mathbb{Z}$ with $-M < n_0 \le n_1 \le N$. Then

$$\sum_{n=-M}^{n_0-1} a_n + \sum_{n=n_0}^{N} a_n = \sum_{n=-M}^{n_1-1} a_n + \sum_{n=n_1}^{N} a_n. \tag{5}$$

Example 2.2. It is trivial that the series $\sum_{n=-\infty}^{+\infty} (1/2)^{|n|}$ has sum 3 in traditional analysis.

Instead, recalling that $\sum_{n=0}^{m} (1/2)^n = 2 - (1/2)^m$, in the new framework we can write

$$\sum_{n=-①}^{①} \left(\frac{1}{2}\right)^{|n|} = -1 + 2\sum_{n=0}^{①} \left(\frac{1}{2}\right)^{n} = 3 - \left(\frac{1}{2}\right)^{① - 1} \tag{6}$$

for such a sum.

In this case the sums, both in the old and in the new setting, give a finite number as result. But we remark again a capital difference: when we write $\sum_{n=-\infty}^{+\infty} (1/2)^{|n|} = 3$, this is **not** an equality in a strict sense, because the writing $\sum_{n=-\infty}^{+\infty} (1/2)^{|n|}$ hides a (double) limit, i.e., it is an approaching process. Moreover, as mentioned immediately before Example 2.1, in the new framework an infinite sum depends from the ending as well as from the initial summand. For example we have

$$\sum_{n=-①}^{①+2} \left(\frac{1}{2}\right)^{|n|} = 3 - 5\left(\frac{1}{2}\right)^{① + 2} \qquad \text{and} \qquad \sum_{n=-2①+1}^{3①^2+2①} \left(\frac{1}{2}\right)^{|n|} = 3 - \left(\frac{1}{2}\right)^{2①} \cdot \left[2 + \left(\frac{1}{2}\right)^{3①^2}\right],$$

which are both different from the result in (6), even if just for infinitesimal quantities in this case.

In general instead, changing the number of elements in a sum we can get different results (as it happens with sums having a finite number of addends), as the following trivial examples show

$$\sum_{n=-①+1}^{①} \frac{1}{①^2} = \frac{2}{①}, \qquad \sum_{n=1}^{①^2} \frac{1}{①^2} = 1 \qquad \text{and} \qquad \sum_{n=-①^2+1}^{①^3} \frac{1}{①^2} = ① + 1,$$

in which we obtain an infinitesimal, finite and infinite number, respectively. This is interesting since it shows that a sum of infinitely many infinitesimals can be finite, infinite or infinitesimal in dependence of the number of summands.

Example 2.3. The double series $\sum_{n=-\infty}^{+\infty} (1/2)^n$ diverges (just) to $+\infty$ in traditional mathematics. Instead, adopting the new paradigm and taking $M = N = ①$, we achieve the well determined result

$$\sum_{n=-①}^{①} \left(\frac{1}{2}\right)^{n} = -1 + \sum_{n=0}^{①} 2^n + \sum_{n=0}^{①} \left(\frac{1}{2}\right)^{n} = 2^{① + 1} - 2^{-①},$$

and similarly we can calculate such a sum if we change the starting or the final point. We remark the important fact that series filed just as "divergent" in traditional analysis, are now sums with a precise (infinite) resulting value, and we can handy work with them in a completely new way.

Example 2.4. Consider $\sum_{n=-\infty}^{+\infty} \text{sgn}(n)$, where sgn( ) denotes the *sign function*[2].

[2]We recall that, by definition, $\text{sgn}(0) = 0$ and $\text{sgn}(x) = x/|x|$ if x is a nonzero real number.

(i) Obviously, this series does not converge in traditional analysis because $\sum\limits_{n=-\infty}^{-1} \mathrm{sgn}(n) + \sum\limits_{n=0}^{+\infty} \mathrm{sgn}(n) = +\infty - (+\infty)$, and so the sum of the two limits is not defined.

(ii) In the new framework, taking $M = N = ①$, the sum $\sum\limits_{n=-①}^{①} \mathrm{sgn}(n)$ is equal to zero. In fact, since $\mathrm{sgn}(0) = 0$, then $\sum\limits_{n=0}^{①} \mathrm{sgn}(n) = ①$ and $\sum\limits_{n=-①}^{-1} \mathrm{sgn}(n) = -①$. Hence $\sum\limits_{n=-①}^{①} \mathrm{sgn}(n) = 0$.

In conclusion, the given series does not converge in traditional analysis and, even more, the sum of the series does not exist at all. Instead, adopting the new computational system, its sum is simply zero. Note, moreover, that also in this case we can easily calculate the sum changing the starting and the final points. For example, we have

$$\sum_{n=-①+2}^{4①^2+①-3} \mathrm{sgn}(n) = \sum_{n=1}^{①-2} \mathrm{sgn}(-n) + \sum_{n=1}^{4①^2+①-3} \mathrm{sgn}(n) = 4①^2 - 1,$$

and if someone wanted to generalize again up to the highest degree, then he would get a formula even more trivial than the previous, because he could just write $\sum\limits_{n=-M}^{N} \mathrm{sgn}(n) = N - M$ where M, N are any elements of $\widehat{\mathbb{N}}$.

Example 2.5. Consider the series **(a)** $\sum\limits_{n=-\infty}^{+\infty} n$, **(b)** $\sum\limits_{n=-\infty}^{+\infty} (n+1)$.

(i) With traditional analysis, the sum of both series is again indeterminate of the kind $+\infty-(+\infty)$, as in the previous example.

(ii) Recalling that $\sum\limits_{n=1}^{①} n = \frac{1}{2} \cdot ① \,(① + 1)$ then, in the new framework by taking $M = N = ①$, the sum **(a)** returns zero and **(b)** yields

$$\sum_{n=-①}^{①} (n+1) = \sum_{n=0}^{①}(n+1) + \sum_{n=-①}^{-1} (n+1) = \frac{(①+1)(①+2)}{2} - \frac{①(①-1)}{2} = 2① + 1.$$

Note that, in fact, the last result agrees with the one just got for **(a)** and with the fact that $\mathbb{Z}$ has $2①+1$ elements.

Example 2.6. Consider the series **(a)** $\sum\limits_{n=-\infty}^{+\infty} |n|$, **(b)** $\sum\limits_{n=-\infty}^{+\infty} |n+1|$, **(c)** $\sum\limits_{n=-\infty}^{+\infty} n^2$, **(d)** $\sum\limits_{n=-\infty}^{+\infty} (n+1)^2$.

(i) In traditional analysis, their sums are $+\infty$ in all four cases. We remark that four different series give the same sum (represented by the symbol $+\infty$) and we cannot distinguish one result from the others.

(ii) With the grossone-based system, by fixing $M = N = ①$, we find the following identities:

(a) $\sum\limits_{n=-①}^{①} |n| = ①^2 + ①$,

(b) $\sum\limits_{n=-①}^{①} |n+1| = ①^2+① + 1$,

(c) $\sum\limits_{n=-①}^{①} n^2 = \dfrac{2①^3 + 3①^2 + ①}{3}$,

(d) $\sum\limits_{n=-①}^{①} (n+1)^2 = \dfrac{2①^3 + 3①^2 + 7① + 3}{3}$,

where for the last two sums we used the well know equality $\sum\limits_{k=1}^{n} k^2 = (2n^3 + 3n^2 + n)/6$. Note, in particular, that the four sums have now four **different** resulting values. So, in this case, we can distinguish the former expressions just by looking at their final outputs.

3. The Z-transform in the grossone framework

The Z-transform is a fundamental tool to process signals. It converts a discrete-time signal, which is a sequence of real or complex numbers, into a complex frequency domain representation (see, for example, [11, 14, 17, 18, 24, 36, 39, 40]).

Definition 3.1. The *Z-transform* of a sequence $\{x(k)\}_{k=-\infty}^{\infty}$ of complex numbers, denoted by $\mathscr{Z}[x(k)]$ or $X(z)$, is defined as the following series

$$\mathscr{Z}[x(k)] = X(z) := \sum_{k=-\infty}^{+\infty} x(k)z^{-k}, \tag{7}$$

in the complex variable z.

Remark 3.1. The Z-transform in the previous form (7) is often referred to as the *bilateral* or *two-sided Z-transform*. In fact, if the sequence $\{x(k)\}_k$ is defined only for nonnegative values of k (or, sometimes, if $x(k) = 0$ for $k < 0$), the power series

$$\mathscr{Z}[x(k)] = X(z) := \sum_{k=0}^{+\infty} x(k)z^{-k} \tag{8}$$

is usually called the *unilateral* or *one-sided Z-transform* of $\{x(k)\}_{k=0}^{\infty}$.

Since the bilateral Z-transform of a sequence is a Laurent series, it exists only for those values of z for which the summation in equation (7) converges in the sense of Definition 2.1. Thus, if we use the following splitting

$$\sum_{k=-\infty}^{+\infty} x(k)z^{-k} = \sum_{k=-\infty}^{-1} x(k)z^{-k} + \sum_{k=0}^{+\infty} x(k)z^{-k}, \tag{9}$$

the doubly-infinite series (7) converges for some $z \in \mathbb{C}$ if and only if both these split series do. The first term in the right side of (9) is a power series and will have a radius of convergence $R \in [0, +\infty]$. The second one can be viewed as a power series in $w = \frac{1}{z}$ and will converge on $|w| < \rho$ for some $\rho \in [0, +\infty]$, i.e. $|z| > r := \frac{1}{\rho}$ (with the obvious conventions if $\rho = 0, +\infty$). Therefore, the series (7) will converge absolutely for all points in the z-plane that lie in the annulus $r < |z| < R$. This region is called the *region of convergence*, briefly, ROC. We observe that the ROC cannot contain any poles and any singularities, and moreover, if $r \geq R$ there is no common region of convergence for the two sums: in the last case, hence, $X(z)$ does not exist anywhere.

Using the new language we are able to analyze with more accuracy the Z-transform of a sequence, and to give a much more satisfactory and precise answer to its existence. Note, first of all, that in the new framework many different infinite numbers can be distinguished, and this means that records (7) and (8), viewed in the new environment, do not describe a single function but there exist many different Z-transforms, bilateral or unilateral, of the same sequence. To choose a particular one it is necessary to fix the number of items and the starting point in the corresponding sum. Hence, in conclusion, we no longer have a single defining expression for the Z-transform function but, for every pair of non-negative integers M, N belonging to the extended set $\widehat{\mathbb{N}}_0$, we pose

$$X(z, -M, N) := \sum_{k=-M}^{N} x(k)z^{-k}. \tag{10}$$

Therefore, for different values of M and N we find different Z-transforms. When $M = N$ we speak of *symmetric* Z-transform (without any assumption on $x(-M)$ and $x(N)$ equal or different from zero) and, to short notations, we also set

$$X(z, N) := X(z, -N, N) = \sum_{k=-N}^{N} x(k)z^{-k} \tag{11}$$

for all $N \in \widehat{\mathbb{N}}_0$.

In the next subsections we will illustrate the new approach to Z-transform by considering some important and illuminating examples, that can be taken as a model in order to adopt the new framework in dealing with Z-transforms. Starting from such examples, it is in fact easy to build several Z-transforms in many other cases, by imitating our constructions and our analysis.

3.1. The Z-transform of the exponential sequence in the new setting

In this and the next subsections we will consider some simple sequences that represent fundamental discrete-time functions in signal theory, and we will determine and study the Z-transforms arising from them through the lens of the grossone system.

Let us take the sequence

$$x(k) = 2^{-k}, \quad k \in \mathbb{Z}; \tag{12}$$

from Equation (9), using the traditional approach, we get that its Z-transform is the function

$$X(z) = \sum_{k=-\infty}^{+\infty} (2z)^{-k} = \sum_{k=-\infty}^{-1} (2z)^{-k} + \sum_{k=0}^{+\infty} (2z)^{-k}. \tag{13}$$

The second term is a geometric series of ratio $\frac{1}{2z}$, its ROC is the range of values of z for which $|z| > \frac{1}{2}$ and

$$\sum_{k=0}^{+\infty} (2z)^{-k} = \frac{2z}{2z-1}. \tag{14}$$

By replacing the index k with $-h$, the first term becomes

$$\sum_{h=1}^{+\infty} (2z)^{h}$$

that is a geometric series of ratio $2z$. The ROC is the range of values of z for which $|z| < \frac{1}{2}$ and

$$\sum_{h=1}^{+\infty} (2z)^{h} = \frac{1}{1-2z} - 1 = \frac{2z}{1-2z}. \tag{15}$$

In both terms, the respective convergence regions do not include the values of z for which $|z| = \frac{1}{2}$; in fact, the value $z = \frac{1}{2}$ is a pole for the obtained functions in (14) and (15). We conclude that, since $r = R = \frac{1}{2}$, the bilateral transform $X(z)$ does not exist anywhere.

In the new language instead, to determine a Z-transform function, it is necessary, as previously said, to fix the number of items through the choice of M, N in (10). For example, if $M = N = ①$ we obtain

$$X(z, ①) = \sum_{k=-①}^{①} (2z)^{-k} = \sum_{k=-①}^{-1} (2z)^{-k} + \sum_{k=0}^{①} (2z)^{-k}. \tag{16}$$

Let us consider $z \neq 0, \frac{1}{2}$; then for the last summation, taking into account the formula (3), we get

$$\sum_{k=0}^{①} (2z)^{-k} = \frac{1 - \left(\frac{1}{2z}\right)^{①+1}}{1 - \frac{1}{2z}} = \frac{2^{①+1} z^{①+1} - 1}{2^{\,①} z^{①}(2z-1)}, \tag{17}$$

which can be also written in the following form

$$\sum_{k=0}^{①} (2z)^{-k} = \frac{2z}{2z-1} - \frac{1}{2z-1} \cdot \left(\frac{1}{2z}\right)^{①}. \tag{18}$$

Note that (14) and (18) differ by the infinitesimal quantity $\frac{1}{2z-1} \cdot \left(\frac{1}{2z}\right)^{①}$.

Analogously, by replacing the index k with $-h$ and using (3) again, we find for the first term in the right part of (16)

$$\sum_{k=-①}^{-1} (2z)^{-k} \; = \; \sum_{h=1}^{①} (2z)^{h} = \frac{1-2^{\,①+1} z^{①+1}}{1-2z} - 1 \; = \; \frac{2z}{1-2z} - \frac{1}{1-2z} \cdot (2z)^{①+1}. \tag{19}$$

Hence, adding (18) and (19), we obtain

$$X(z, ①) \; = \; \frac{1}{2z-1} \cdot \left((2z)^{①+1} - (2z)^{-①}\right),$$

for all $z \neq 0, \frac{1}{2}$. Moreover, note that we can also calculate the Z-transform in the pole $z = \frac{1}{2}$ of traditional analysis for both (14) and (15); in fact, from (13), we can directly compute

$$X\left(\frac{1}{2}, ①\right) \; = \; \sum_{k=-①}^{①} 1 \; = \; 1 + 2\sum_{k=1}^{①} 1 \; = \; 2① + 1.$$

As already mentioned, taking for instance other infinite values in the place of ① in (10), we get different symmetric Z-transform of the sequence (12). For example, if we choose $N = \frac{①}{2}$, by easy computations similar to the previous ones, we find

$$X\left(z, \frac{①}{2}\right) \; = \; \begin{cases} \dfrac{1}{2z-1} \cdot \left((2z)^{\,1+①/2} - (2z)^{-①/2}\right) & \text{if} \quad z \neq 0, \; \frac{1}{2}; \\ ① + 1 & \text{if} \quad z = \frac{1}{2}. \end{cases}$$

The results above can be generalized to the sequence $x(k) = a^k$, $k \in \mathbb{Z}$, where a is any real or complex number different from zero. We also recall that the numbers M, N in (10) and (11) can be greater than ① too; for example, if $N = ①^2$ the correspondent symmetric Z-transform for the sequence $\{a^k\}_k$ is

$$X\left(z, ①^2\right) \; = \; \begin{cases} \dfrac{a}{z-a} \cdot \left[\left(\dfrac{z}{a}\right)^{①^2+1} - \left(\dfrac{z}{a}\right)^{-①^2}\right] & \text{if} \quad z \neq 0, \; a; \\ 2①^2 + 1 & \text{if} \quad z = a. \end{cases}$$

Furthermore, we can also take a equal to an infinite number or an infinitesimal number. For example, if $a = 1/①$ and $N = ①/2$, similar computations yield

$$X\left(z, \frac{①}{2}\right) \; = \; \begin{cases} \dfrac{1}{① z - 1} \cdot \left[(① z)^{1+①/2} - (① z)^{-①/2}\right] & \text{if} \quad z \neq 0, \; 1/①; \\ ① + 1 & \text{if} \quad z = 1/①. \end{cases}$$

Hence, in conclusion, we can state the following proposition that generalizes the previous formulas to all finite or infinite non-negative integers $M, N \in \widehat{\mathbb{N}}_0$.

Proposition 3.1. *Let $M, N \in \widehat{\mathbb{N}}_0$ and $a \in \mathbb{C} - \{0\}$. If we set $x(k) = a^k$ for every $k \in \widehat{\mathbb{Z}}$ with $-M \leq k \leq N$, then*

$$X\,(z, -M, N) \; = \; \begin{cases} \dfrac{a}{z-a} \cdot \left[\left(\dfrac{z}{a}\right)^{M+1} - \left(\dfrac{z}{a}\right)^{-N}\right] & \textit{if} \quad z \neq 0, \; a; \\ M + N + 1 & \textit{if} \quad z = a. \end{cases}$$

3.2. The Z-transform of the unit ramp sequence

A pure mathematician would refer to the sequence $x(k) = k,\ k \in \mathbb{Z}$, simply as the identity map on the integers, instead, signal theory specialists often call $x(k)$ (and some similar others) the *unit ramp sequence*. The classical Z-transform of $\{x(k)\}_k$ yields the complex function

$$X(z) \;=\; \sum_{k=-\infty}^{+\infty} kz^{-k} \;=\; \sum_{k=-\infty}^{-1} kz^{-k} + \sum_{k=0}^{+\infty} kz^{-k}. \tag{20}$$

The ROC of the last summation is the range of values of z for which $|z| > 1$ and it is also known that it holds the following identity (which can be also easily deduced from the formula (23) below, as we shall see shortly)

$$\sum_{k=0}^{+\infty} kz^{-k} \;=\; \frac{z}{(z-1)^2}. \tag{21}$$

Instead, the ROC of the first summation in the right part of (20) is the open unitary disc $D_1(0) = \{z \in \mathbb{C} \,:\, |z| < 1\}$ and we have the classical formula

$$\sum_{k=-\infty}^{-1} kz^{-k} \;=\; \frac{-z}{(z-1)^2}.$$

As in the previous example, the Z-transform does not exist again in traditional sense, and both terms in the right hand side of (20) become infinite in the common pole $z = 1$.

Now we go to consider the Z-transform of our sequence in the grossone-based framework and we take $N = ①$ in (11); so we have

$$X\,(z, ①) \;=\; \sum_{k=-①}^{①} kz^{-k} \;=\; \sum_{k=-①}^{-1} kz^{-k} + \sum_{k=0}^{①} kz^{-k}. \tag{22}$$

Multiplying the last summand in the previous equation by $1 - z^{-1}$, we find

$$\begin{aligned}
(1 - z^{-1}) \sum_{k=0}^{①} kz^{-k} \;&=\; (1 - z^{-1}) \cdot \left(\frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots + \frac{① - 1}{z^{①-1}} + \frac{①}{z^{①}} \right) \\
&=\; \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \ldots + \frac{① - 1}{z^{①-1}} + \frac{①}{z^{①}} \; - \left(\frac{1}{z^2} + \frac{2}{z^3} + \ldots + \frac{① - 1}{z^{①}} + \frac{①}{z^{①+1}} \right) \\
&=\; \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots + \frac{1}{z^{①}} - \frac{①}{z^{①+1}} \\
&=\; \frac{z^{①+1} - 1}{z^{①}(z - 1)} - 1 - \frac{①}{z^{①+1}} \\
&=\; \frac{z^{①+1} - (① + 1)z + ①}{z^{①+1}(z - 1)},
\end{aligned}$$

and dividing by $1 - z^{-1}$, it yields

$$\sum_{k=0}^{①} kz^{-k} \;=\; \frac{z^{①+1} - (① + 1)\,z + ①}{z^{①}(z - 1)^2} \;=\; \frac{z}{(z - 1)^2} \;+\; \frac{-\,(① + 1)\,z + ①}{(z - 1)^2} \cdot \frac{1}{z^{①}}. \tag{23}$$

Note that, if we overlook the last summand (which represent an infinitesimal quantity for every fixed $z \neq 0$) in the previous equation, we recover easily the formula (21), as we expected.

Let us consider the first sum in the right hand side of (22): by replacing the index k with $-h$, we obtain

$$\sum_{k=-\text{①}}^{-1} kz^{-k} \;=\; \sum_{h=1}^{\text{①}} -hz^{h} \;=\; -\left(z+2z^2+3z^3+\ldots+(\text{①}-1)z^{\text{①}-1}+\text{①}\,z^{\text{①}}\right), \tag{24}$$

and multiplying by $z-1$, we get

$$\begin{aligned}(z-1)\sum_{k=-\text{①}}^{-1} kz^{-k} \;&=\; -\left(z^2+2z^3+3z^4+\ldots+\text{①}\,z^{\text{①}+1}\right)\;+z+2z^2+3z^3+\ldots+\text{①}\,z^{\text{①}}\\ &=\; z+z^2+z^3+z^4+\ldots+z^{\text{①}}\;-\text{①}\,z^{\text{①}+1}\\ &=\; \frac{1-z^{\text{①}+1}}{1-z}-1-\text{①}\,z^{\text{①}+1}\\ &=\; \frac{-z^{\text{①}+1}+z-\text{①}\,z^{\text{①}+1}+\text{①}\,z^{\text{①}+2}}{1-z}.\end{aligned}$$

Hence, dividing by $z-1$, we obtain

$$\sum_{k=-\text{①}}^{-1} kz^{-k} \;=\; \frac{z^{\text{①}+1}-z+\text{①}\,z^{\text{①}+1}-\text{①}\,z^{\text{①}+2}}{(1-z)^2} \;=\; -\frac{z}{(1-z)^2}\;+\frac{-\text{①}\,z+\text{①}+1}{(1-z)^2}\cdot z^{\text{①}+1}. \tag{25}$$

Finally, adding (23) and (25), we find the following expression for the Z-transform

$$X(z,\text{①}) \;=\; \frac{-\text{①}\,z+\text{①}+1}{(z-1)^2}\cdot z^{\text{①}+1}+\frac{-(\text{①}+1)\,z+\text{①}}{(z-1)^2}\cdot z^{-\text{①}},$$

for all $z \neq 0,\ 1$. Moreover note that, as in the previous example, we can compute the Z-transform $X(z,\text{①})$ also in the point $z=1$, obtaining

$$X(1,\text{①}) \;=\; \sum_{k=-\text{①}}^{\text{①}} k \;=\; -\sum_{k=1}^{\text{①}} k+\sum_{k=1}^{\text{①}} k \;=\; 0.$$

Therefore, in conclusion, we can state the following more general claim as we did in Proposition 3.1.

Proposition 3.2. *Let $M,N \in \widehat{\mathbb{N}}_0$ and $x(k)=k$ for $-M \leq k \leq N$. Then*

$$X(z,-M,N) \;=\; \begin{cases} \dfrac{-Mz+M+1}{(z-1)^2}\cdot z^{M+1}+\dfrac{-(N+1)z+N}{(z-1)^2}\cdot z^{-N} & \textit{if}\quad z\neq 0,\ 1;\\[2ex] \dfrac{(N-M)(N+M+1)}{2} & \textit{if}\quad z=1.\end{cases}$$

3.3. The Z-transform of the parabola sequence

Let us consider the sequence $x(k)=k^2,\ k\in\mathbb{Z}$. The classical Z-transform gives the series

$$X(z)=\sum_{k=-\infty}^{+\infty} k^2z^{-k}=\sum_{k=-\infty}^{-1} k^2z^{-k}+\sum_{k=0}^{+\infty} k^2z^{-k}. \tag{26}$$

The ROC of the last summation is the range of values of z for which $|z|>1$, and it is also well known that it holds the following identity (which can be also viewed as consequence of formula (29) below)

$$\sum_{k=0}^{+\infty} k^2z^{-k}=\frac{z(z+1)}{(z-1)^3}. \tag{27}$$

Instead, the ROC of the first summation in the right hand side of (26) is the unitary open disc $D_1(0)$ and, from (27), we have immediately the formula

$$\sum_{k=-\infty}^{-1} k^2 z^{-k} = \frac{z(z+1)}{(1-z)^3}.$$

Also in this case, the bilateral Z-transform does not exist and both summations in the right part of (26) become infinite in the common pole $z = 1$.

Now we will see as, using the new grossone-based system, things change, once again, deeply. If we consider, for example, the Z-transform (11) with $N = ①$, we can write

$$X(z, ①) \;=\; \sum_{k=-①}^{①} k^2 z^{-k} \;=\; \sum_{k=-①}^{-1} k^2 z^{-k} + \sum_{k=0}^{①} k^2 z^{-k}. \tag{28}$$

Let $z \neq 0, 1$ and note that

$$\begin{aligned}
\left(1 - z^{-1}\right) \sum_{k=0}^{①} k^2 z^{-k} &= \left(1 - z^{-1}\right) \cdot \left[\frac{1}{z} + \frac{2^2}{z^2} + \frac{3^2}{z^3} + \frac{4^2}{z^4} + \ldots + \frac{①^2}{z^{①}}\right] \\
&= \left[\frac{1}{z} + \frac{2^2}{z^2} + \frac{3^2}{z^3} + \frac{4^2}{z^4} + \ldots + \frac{①^2}{z^{①}}\right] - \left[\frac{1}{z^2} + \frac{2^2}{z^3} + \frac{3^2}{z^4} + \frac{4^2}{z^5} + \ldots + \frac{①^2}{z^{①+1}}\right] \\
&= \left[\frac{1}{z} + \frac{3}{z^2} + \frac{5}{z^3} + \frac{7}{z^4} + \frac{9}{z^5} + \ldots + \frac{2① - 1}{z^{①}}\right] - \frac{①^2}{z^{①+1}} \\
&= \left[\frac{2}{z} + \frac{4}{z^2} + \frac{6}{z^3} + \frac{8}{z^4} + \ldots + \frac{2①}{z^{①}}\right] - \left[\frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots + \frac{1}{z^{①}}\right] - \frac{①^2}{z^{①+1}} \\
&= 2 \cdot \left[\frac{z}{(z-1)^2} + \frac{-(① + 1)z + ①}{(z-1)^2} \cdot \frac{1}{z^{①}}\right] - \frac{1 - (1/z)^{①+1}}{1 - 1/z} - \frac{①^2}{z^{①+1}} \qquad \left(\text{using (23)}\right) \\
&= \frac{z+1}{(z-1)^2} + \frac{-\left(①^2 + 2① + 1\right) z^2 + \left(2①^2 + 2① - 1\right) z - ①^2}{(z-1)^2 \cdot z^{①+1}},
\end{aligned}$$

then, dividing by $1 - z^{-1}$, we get

$$\sum_{k=0}^{①} \frac{k^2}{z^k} \;=\; \frac{z(z+1)}{(z-1)^3} + \frac{-\left(①^2 + 2① + 1\right) z^2 + \left(2①^2 + 2① - 1\right) z - ①^2}{(z-1)^3} \cdot \frac{1}{z^{①}}. \tag{29}$$

Let now consider the first summation in the right side of (28); by easy computations we have

$$\begin{aligned}
\sum_{k=-①}^{-1} \frac{k^2}{z^k} &= \sum_{k=1}^{①} k^2 \cdot z^k \;=\; \sum_{k=1}^{①} \frac{k^2}{w^k} \qquad \left(\text{placing } w = \frac{1}{z}\right) \\
&= \frac{w(w+1)}{(w-1)^3} + \frac{-(① + 1)^2 w^2 + \left(2①^2 + 2① - 1\right) w - ①^2}{(w-1)^3 \cdot w^{①}} \qquad \left(\text{using (29)}\right),
\end{aligned}$$

from which, using again the substitution $w = 1/z$, we can conclude that

$$\sum_{k=-①}^{-1} k^2 z^{-k} \;=\; \frac{z(z+1)}{(1-z)^3} + \frac{①^2 z^2 - \left(2①^2 + 2① - 1\right) z + (① + 1)^2}{(1-z)^3} \cdot z^{①+1} \tag{30}$$

holds for any $z \neq 0, 1$. In conclusion, adding (29) and (30), we get the following formula for the symmetric Z-transform (with $N = ①$) of the sequence $\{k^2\}_k$

$$X(z, ①) = \frac{①^2 z^2 - \left(2①^2 + 2① - 1\right) z + (① + 1)^2}{(1-z)^3} \cdot z^{①+1} + \frac{\left(①^2 + 2① + 1\right) z^2 - \left(2①^2 + 2① - 1\right) z + ①^2}{(1-z)^3} \cdot z^{-①},$$

where z can be any complex number just different from 0, 1. On the other hand, if $z = 1$, it is simple to compute

$$X(1, ①) = \sum_{k=-①}^{①} k^2 = 2 \sum_{k=1}^{①} k^2 = \frac{①(① + 1)(2① + 1)}{3} = \frac{2}{3}①^3 + ①^2 + \frac{1}{3}①,$$

where we used the well known formula

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

which holds, in the new framework, for any positive finite and infinite integer N.

Finally, if the reader wanted a formula for $X(z, -M, N)$ in full generality, then he could easily prove the following

Proposition 3.3. *Let* $M, N \in \widehat{\mathbb{N}}_0$ *and* $x(k) = k^2$ *for* $-M \leq k \leq N$. *Then*

$$X(z, -M, N) = \begin{cases} \frac{1}{(1-z)^3} \Big[\Big(M^2 z^2 - \left(2M^2 + 2M - 1\right) z + (M+1)^2 \Big) \cdot z^{M+1} \\ \qquad + \Big((N+1)^2 z^2 - \left(2N^2 + 2N - 1\right) z + N^2 \Big) \cdot z^{-N} \Big] & \text{if } \; z \neq 0,\ 1; \\ \frac{M(M+1)(2M+1) + N(N+1)(2N+1)}{6} & \text{if } \; z = 1. \end{cases}$$

4. Some properties of the new grossone-based Z-transform

In this section we will give some properties of the Z-transform in the grossone framework. We emphasize that what follows should not be understood as an exhaustive exposition, but only as the first more abstract examples, always introductory, in the same spirit and with the same purposes as the rest of the paper. In particular we show as some well know properties of the classical Z-transform continue to hold, but in a revisited manner and adopting the appropriate variations dictated by the change of the setting. We moreover point out that we preferred, in the following, simple and easily legible claims, rather than formulating the statements in the maximum possible generality.

As regards the notations, in agreement with those of the previous section, $\{x(k)\}_k$, $\{x_n(k)\}_k$ ($n \in \mathbb{N}$) and $\{y(k)\}_k$ denote complex sequences on the integers (more precisely, they are functions from $\mathbb{Z}$ or from subsets of $\widehat{\mathbb{Z}}$, to $\mathbb{C}$ or $\widehat{\mathbb{C}}$) whose Z-transforms will be indicated by $X(z) = X(z, -M, N)$, $X_n(z) = X_n(z, -M, N)$ and $Y(z) = Y(z, -M, N)$, respectively.

Proposition 4.1 (Time shifting). *Let* $\{x(k)\}_k$ *be any complex sequence and* $l \in \mathbb{Z}$ *a fixed integer. Posing* $y(k) = x(k-l)$, *we have*

$$Y(z, -M, N) = z^{-l} \cdot X(z, -M-l, N-l) \tag{31}$$

for all $M, N \in \widehat{\mathbb{N}}_0$ *with* $-M \leq l \leq N$ *(and all* $z \in \mathbb{C} - \{0\}$*).*

Proof. Using the definition in (10) and the substitution $h = k - l$, we obtain

$$\begin{aligned} Y(z, -M, N) &= \sum_{k=-M}^{N} y(k) z^{-k} = \sum_{k=-M}^{N} x(k-l) z^{-k} \\ &= \sum_{h=-M-l}^{N-l} x(h) z^{-h-l} = z^{-l} \cdot \sum_{h=-M-l}^{N-l} x(h) z^{-h} = z^{-l} \cdot X(z, -M-l, N-l), \end{aligned}$$

for all $M, N \in \widehat{\mathbb{N}}_0$ with the only conditions $M + l \geq 0$ and $N - l \geq 0$, that is $-M \leq l \leq N$.[3] □

We can give different formulations of Proposition 4.1 and of the next ones in this section; for instance, we can state it as follows

$$\text{“Let } M, N \in \widehat{\mathbb{N}}_0,\ \{x(k)\}_{k=-M}^{N} \text{ be a complex sequence and } l \in \mathbb{Z} \text{ a fixed integer with } -M \leq l \leq N. \text{ If we pose } y(k) = x(k-l), \text{ then } Y(z, -M, N) = z^{-l} \cdot X(z, -M-l, N-l) \text{ for all } z \in \mathbb{C} - \{0\}.\text{”} \tag{32}$$

Note that (32) provides a more precise writing for the hypotheses on the sequence $\{x(k)\}_k$ and, in particular, regarding its definition domain. Instead, in the following we continue to express such hypotheses like in Proposition 4.1 to emphasize that, in concrete examples, the sequence $x(k)$ is given not just for k between $-M$ and N but, in almost all cases, for all $k \in \widehat{\mathbb{Z}}$, and we can distinguish infinitely many Z-transforms by changing M and N (see Section 3).

Proposition 4.2 (**Time reversal**). *Let $\{x(k)\}_k$ be any complex sequence and pose $y(k) = x(-k)$. Then*

$$Y(z, -M, N) = X(z^{-1}, -N, M) \tag{33}$$

for all $M, N \in \widehat{\mathbb{N}}_0$.

Proof. Using the substitution $h = -k$, we get

$$\begin{aligned} Y(z, -M, N) &= \sum_{k=-M}^{N} y(k) z^{-k} = \sum_{k=-M}^{N} x(-k) z^{-k} \\ &= \sum_{h=-N}^{M} x(h) z^{h} = \sum_{h=-N}^{M} x(h) \left(z^{-1}\right)^{-h} = X\left(z^{-1}, -N, M\right), \end{aligned}$$

for all $M, N \in \widehat{\mathbb{N}}_0$ and all $z \in \mathbb{C} - \{0\}$. □

Proposition 4.3 (**Change of scale**). *Let $\{x(k)\}_k$ be a complex sequence, a any nonzero element of $\mathbb{C}$ and pose $y(k) = a^k x(k)$. Then*

$$Y(z, -M, N) = X\left(a^{-1} z, -M, N\right) \tag{34}$$

for all $M, N \in \widehat{\mathbb{N}}_0$.

Proof. Note that

$$Y(z, -M, N) = \sum_{k=-M}^{N} y(k) z^{-k} = \sum_{k=-M}^{N} a^k x(k) z^{-k} = \sum_{h=-M}^{N} x(k) \left(a^{-1} z\right)^{-k} = X(a^{-1} z, -M, N),$$

for all $M, N \in \widehat{\mathbb{N}}_0$ and all $z \in \mathbb{C} - \{0\}$. □

Proposition 4.4 (**Differentiation**). *Let $\{x(k)\}_k$ be a complex sequence and $y(k) = kx(k)$. Then*

$$Y(z, -M, N) = -z \frac{\mathrm{d}X(z, -M, N)}{\mathrm{d}z} \tag{35}$$

for all $M, N \in \widehat{\mathbb{N}}_0$.

[3] Note that the requests $M + l \geq 0$ and $N - l \geq 0$ depend just from the convenience to have $M, N \geq 0$ in the definition of $X(z, -M, N)$ in (10).

Proof. Recalling the natural definition of differentiation of a formal power series, well known in abstract algebraic contexts, we immediately have

$$\begin{aligned} Y(z,-M,N) &= \sum_{k=-M}^{N} y(k)z^{-k} = \sum_{k=-M}^{N} kx(k)z^{-k} = -z\sum_{h=-M}^{N} -k\,x(k)z^{-k-1} \\ &= -z\sum_{h=-M}^{N} \frac{\mathrm{d}\left(x(k)z^{-k}\right)}{\mathrm{d}z} = -z\frac{\mathrm{d}}{\mathrm{d}z}\left(\sum_{h=-M}^{N} x(k)z^{-k}\right) = -z\frac{\mathrm{d}X(z,-M,N)}{\mathrm{d}z}, \end{aligned}$$

for all $M, N \in \widehat{\mathbb{N}}_0$. □

A generalization of the previous proposition can be stated as expected in the following form.

Proposition 4.5 (***n*-time differentiation**). *Let $\{x(k)\}_k$ be a complex sequence and let $x_n(k) = k(k-1)\ldots(k+n-1)\cdot x(k)$ for any $n \in \mathbb{N}$. Then*

$$X_n(z,-M,N) = (-1)^n z^n \frac{\mathrm{d}^n X(z,-M,N)}{\mathrm{d}z^n} \tag{36}$$

for all $M, N \in \widehat{\mathbb{N}}_0$.

Proof. By means of simple calculations we in fact have

$$\begin{aligned} (-1)^n z^n \frac{\mathrm{d}^n X(z,-M,N)}{\mathrm{d}z^n} &= (-1)^n z^n \frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(\sum_{k=-M}^{N} x(k)z^{-k}\right) = (-1)^n z^n \sum_{k=-M}^{N} x(k)\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left(z^{-k}\right) \\ &= (-1)^n z^n \sum_{k=-M}^{N} x(k)\,(-k)(-k-1)\ldots(-k-(n-1))\,z^{-k-n} \\ &= (-1)^{2n} \sum_{k=-M}^{N} x(k)\,k(k+1)\ldots(k+(n-1))\,z^{-k} \\ &= \sum_{k=-M}^{N} x_n(k)\,z^{-k} = X_n(z,-M,N) \end{aligned}$$

for all $z \in \mathbb{C} - \{0\}$. □

Propositions 4.1–4.5 are examples where the Z-transform keeps the same shape both over $\mathbb{Z}$ and over $\widehat{\mathbb{Z}}$. On the contrary, the following propositions provide instances in which the Z-transform over $\widehat{\mathbb{Z}}$ will have instead a progressively different and more complex writing than the usual over $\mathbb{Z}$. The reason of this phenomenon is soon clear: it depends mainly on the existence of an upper and lower bound (whether finite or infinite) for the index of the sequence $\{x(k)\}_k$, which prevents, as we shall see, the possibility of operating some easy and common simplifications to which algebra and mathematical analysis have accustomed us, since the study of numerical or formal series began.

Proposition 4.6 (**Time difference**). *Let $\{x(k)\}_k$ be a complex sequence and $y(k) = x(k) - x(k-1)$. Then*

$$Y(z,-M,N) = \left(1 - z^{-1}\right)X(z,-M,N) - x(-M-1)z^{M} + x(N)z^{-N-1} \tag{37}$$

for all $M, N \in \widehat{\mathbb{N}}_0$.

Proof. Let $M, N \in \widehat{\mathbb{N}}_0$ be fixed numbers and using the substitution $h = k - 1$, we have

$$\begin{aligned}
Y(z,-M,N) &= \sum_{k=-M}^{N} y(k) z^{-k} = \sum_{k=-M}^{N} (x(k) - x(k-1)) z^{-k} \\
&= \sum_{k=-M}^{N} x(k) z^{-k} - z^{-1} \sum_{k=-M}^{N} x(k-1) z^{-(k-1)} \\
&= \sum_{k=-M}^{N} x(k) z^{-k} - z^{-1} \left[\sum_{h=-M}^{N} \left(x(h)\, z^{-h} \right) + x(-M-1) z^{M+1} - x(N) z^{-N} \right] \\
&= \left(1 - z^{-1}\right) X(z,-M,N) - x(-M-1) z^{M} + x(N) z^{-N-1}
\end{aligned}$$

for all $z \in \mathbb{C} - \{0\}$. □

While equations (31), (33), (34), (35) and (36) do not present substantial differences, equation (37) contains two extra terms with respect to the usual case over $\mathbb{Z}$ which is commonly written as $Y(z) = \left(1 - z^{-1}\right) X(z)$. The next proposition, concerning *time accumulation*, shows how such divergences can also increase greatly: for this purpose, if we denote by $X(z)$ and $Y(z)$ the respective ordinary Z-transforms of the sequences $\{x(k)\}_{k\in\mathbb{Z}}$ and $\{y(k)\}_{k\in\mathbb{Z}}$, where $y(k) = \sum_{h=-\infty}^{k} x(h)$, then we recall that the relation between them can be expressed as follows

$$Y(z) = \frac{1}{1 - z^{-1}} X(z). \tag{38}$$

Proposition 4.7 (Time accumulation). *Let $M, N \in \widehat{\mathbb{N}}_0$ and $\{x(k)\}_k$ be a complex sequence. If we pose $y(k) = \sum_{h=-M}^{k} x(h)$ for all $-M \le k \le N$, then*

$$Y(z,-M,N) = \sum_{h=0}^{N+M} z^{-h} X(z,-M,N-h) \tag{39}$$

for all $z \in \mathbb{C} - \{0\}$.

Proof. Note that

$$\begin{aligned}
Y(z,-M,N) &= \sum_{k=-M}^{N} y(k) z^{-k} = \sum_{k=-M}^{N} \sum_{h=-M}^{k} x(h)\, z^{-k} \\
&= [x(-M)] z^{M} + [x(-M+1) + x(-M)] z^{M-1} + \ldots \\
&\quad \ldots + [x(N) + x(N-1) + \ldots + x(-M)] z^{-N} \\
&= \left[x(-M) z^{M} + x(-M+1) z^{M-1} + \ldots + x(N) z^{-N} \right] + z^{-1} \left[x(-M) z^{M} \right. \\
&\quad \left. + x(-M+1) z^{M-1} + \ldots + x(N-1) z^{-N+1} \right] + \ldots\ldots + z^{-(N+M)} \left[x(-M) z^{M} \right] \\
&= X(z,-M,N) + z^{-1} X(z,-M,N-1) + \ldots + z^{-(N+M)} X(z,-M,-M) \\
&= \sum_{h=0}^{N+M} z^{-h} X(z,-M,N-h)\,,
\end{aligned}$$

hence formula (39) is proved. □

We close the section with a last example about convolution which requires a suitable definition of *bounded convolution* to recover a property similar to the following well known one

$$\mathscr{Z}\left[(x_1 * x_2)(k)\right] \;=\; X_1(z)\cdot X_2(z) \tag{40}$$

for sequences on the ordinary integers.

Definition 4.1. Let $\{x_1(k)\}_k$, $\{x_2(k)\}_k$ be two complex sequences defined on $\widehat{\mathbb{Z}}$; for any $M, N \in \widehat{\mathbb{N}}_0$ we define a *bounded convolution product* (depending from M, N) by setting, for any $k \in \widehat{\mathbb{Z}}$ with $-2M \le k \le 2N$,

$$(x_1 *_{-M,N} x_2)(k) \;:=\; \sum_{h=(-M)\vee(-N+k)}^{N\wedge(M+k)} x_1(h)\cdot x_2(k-h), \tag{41}$$

where $a \wedge b$ and $a \vee b$ denote $\min\{a, b\}$ and $\max\{a, b\}$, respectively.

Proposition 4.8 (Convolution). *Let $\{x_1(k)\}_k$, $\{x_2(k)\}_k$ be two complex sequences (over $\widehat{\mathbb{Z}}$) and let $M, N \in \widehat{\mathbb{N}}_0$. If we set $y(k) = (x_1 *_{-M,N} x_2)(k)$ for every $k \in \widehat{\mathbb{Z}}$ with $-2M \le k \le 2N$, then*

$$Y(z, -2M, 2N) \;=\; X_1(z, -M, N)\cdot X_2(z, -M, N). \tag{42}$$

Proof. Note that the following equivalence is immediate when $m = k - h$

$$\begin{cases} -2M \le k \le 2N \\ (-M)\vee(-N+k) \;\le\; h \;\le\; N\wedge(M+k) \end{cases} \quad\Leftrightarrow\quad \begin{cases} -M \le m \le N \\ -M \le h \le N \end{cases}, \tag{43}$$

therefore, using in order (41), the substitution $m = k - h$ and (43), we obtain

$$\begin{aligned}
Y(z, -2M, 2N) &= \sum_{k=-2M}^{2N} y(k)\,z^{-k} \;=\; \sum_{k=-2M}^{2N} (x_1 *_{-M,N} x_2)(k)\,z^{-k} \\
&= \sum_{k=-2M}^{2N}\ \sum_{h=(-M)\vee(-N+k)}^{N\wedge(M+k)} x_1(h)\cdot x_2(k-h)\cdot z^{-k} \\
&= \sum_{k=-2M}^{2N}\ \sum_{h=(-M)\vee(-N+k)}^{N\wedge(M+k)} x_1(h)\cdot x_2(k-h)\cdot z^{-h}z^{-(k-h)} \\
&= \sum_{m=-M}^{N}\ \sum_{h=-M}^{N} x_1(h)\cdot x_2(m)\cdot z^{-h}z^{-m} \\
&= \left(\sum_{h=-M}^{N} x_1(h)\,z^{-h}\right)\cdot\left(\sum_{m=-M}^{N} x_2(m)\,z^{-m}\right) \\
&= X_1(z, -M, N)\cdot X_2(z, -M, N).
\end{aligned}$$

□

5. Conclusion

The indispensable and ubiquitous use of the symbol ∞ by classical mathematical analysis, very often does not possess a sufficiently high accuracy when one deals with expressions involving numbers which tend to become rapidly infinite or infinitesimal, like in sequences and series. In particular, traditional mathematics gives poor answers when one deals with divergent or irregular series. In this paper we have shown that the recently proposed grossone-based methodology can be usefully applied to compute the result of doubly infinite sums and to study, from a new point of view, the Z-transform of a sequence of complex numbers as well.

After some preparatory machinery and preliminary examples given in Section 2, and after the main definitions of Section 3, we have conducted a thorough and detailed analysis of three particularly representative and enlightening examples, that come from the exponential, the unit ramp and the parabola sequences (see Subsection 3.1, 3.2 and 3.3, respectively). Starting from them, it is easy to apply our methods and this kind of ideas to many other cases provided from real, complex or even more general type of sequences.

We remark also the important fact, revealed by our investigation and concretely shown in our examples, that the Z-transform, in the new setting, is not a single function but many different functions, in dependence on the chosen infinite numbers for the beginning and the end of the sum representing the Z-transform itself. Moreover, in classical analysis, the Z-transform exists only in the convergence region (ROC) of both sides of the double-ended series (9), and it cannot contain any poles and any singularities. Instead, in the new framework we can compute the Z-transform (in theory) for all complex numbers, and also in the poles and in the singularities of the emerging obtained functions.

Lastly, in Section 4 we have shown how some classic properties of the Z-transform, while being repurposable in substance in the new context, require in the form, more or less invasive changes depending on the case.

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