Counting systems and the First Hilbert problem

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Abstract

The First Hilbert problem is studied in the paper by applying two instruments: a new methodology distinguishing in mathematical objects and mathematical languages used to describe these objects; and a new numeral system allowing one to express different infinite numbers and to use these numbers for measuring infinite sets. Several counting systems are taken into consideration. It is emphasized in the paper that different mathematical languages can describe mathematical objects (in particular, sets and the number of their elements) with different accuracies. The traditional and the new approaches are compared and discussed.

Key Words: The First Hilbert problem; numeral systems; Pirahã counting system; relativity of mathematical languages; infinite numbers.

1 Introduction

The First Hilbert problem belongs to the list of famous 23 mathematical problems that have been announced by David Hilbert during his lecture delivered before the International Congress of Mathematicians at Paris in 1900 (see [8]). The problem has the following formulation.

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Cantor's problem of the cardinal number of the continuum

Two systems, i.e., two assemblages of ordinary real numbers or points, are said to be (according to Cantor) equivalent or of equal cardinal number, if they can be brought into a relation to one another such that to every number of the one assemblage corresponds one and only one definite number of the other. The investigations of Cantor on such assemblages of points suggest a very plausible theorem, which nevertheless, in spite of the most strenuous efforts, no one has succeeded in proving. This is the theorem:

Every system of infinitely many real numbers, i.e., every assemblage of numbers (or points), is either equivalent to the assemblage of natural integers, $1, 2, 3, \dots$ or to the assemblage of all real numbers and therefore to the continuum, that is, to the points of a line; as regards equivalence there are, therefore, only two assemblages of numbers, the countable assemblage and the continuum.

From this theorem it would follow at once that the continuum has the next cardinal number beyond that of the countable assemblage; the proof of this theorem would, therefore, form a new bridge between the countable assemblage and the continuum.

Let me mention another very remarkable statement of Cantor's which stands in the closest connection with the theorem mentioned and which, perhaps, offers the key to its proof. Any system of real numbers is said to be ordered, if for every two numbers of the system it is determined which one is the earlier and which the later, and if at the same time this determination is of such a kind that, if a is before b and b is before c, then a always comes before c. The natural arrangement of numbers of a system is defined to be that in which the smaller precedes the larger. But there are, as is easily seen infinitely many other ways in which the numbers of a system may be arranged.

If we think of a definite arrangement of numbers and select from them a particular system of these numbers, a so-called partial system or assemblage, this partial system will also prove to be ordered. Now Cantor considers a particular kind of ordered assemblage which he designates as a well ordered assemblage and which is characterized in this way, that not only in the assemblage itself but also in every partial assemblage there exists a first number. The system of integers $1, 2, 3, \dots$ in their natural order is evidently a well ordered assemblage. On the other hand the system of all real numbers, i.e., the continuum in its natural order, is evidently not well ordered. For, if we think of the points of a segment of a straight line, with its initial point excluded, as our partial assemblage, it will have no first element.

> The question now arises whether the totality of all numbers may not be arranged in another manner so that every partial assemblage may have a first element, i.e., whether the continuum cannot be considered as a well ordered assemblage - a question which Cantor thinks must be answered in the affirmative. It appears to me most desirable to obtain a direct proof of this remarkable statement of Cantor's, perhaps by actually giving an arrangement of numbers such that in every partial system a first number can be pointed out.

Thus, this problem asks about two questions: (i) Is there a set whose cardinality is strictly between that of the natural numbers and that of the real numbers? (ii) Can the continuum of real numbers be considered a well ordered set?

The question (i) has been stated by Cantor and its answer 'no' has been advanced as the Continuum Hypothesis. Gödel (see [5]) proved in 1938 that the generalized continuum hypothesis is consistent relative to Zermelo–Fraenkel set theory. In 1963, Paul Cohen (see [3]) showed that its negation is also consistent. These results are not universally accepted as the final solution to the Continuum Hypothesis and this area remains an active topic of contemporary research (see, e.g., [17, 18]). The modern point of view on the second question tells that it is related to Zermelo's axiom of choice that was demonstrated to be independent of all other axioms in set theory, so there appears to be no universally valid solution to the question (ii) either.

In this paper, we look at the First Hilbert problem using a new approach introduced in [13]–[16]. The point of view on infinite and infinitesimal quantities proposed in it uses two methodological ideas borrowed from the modern Physics: relativity and interrelations between the object of an observation and the tool used for this observation. The latter is applied to mathematical languages used to describe mathematical objects and the objects themselves.

In the next section, we give a description of the methodological principles of the new approach to the interplay between the counting systems and the objects these systems express (a comprehensive introduction and examples of its usage can be found in [14, 15]).

2 Counting systems and the relativity of mathematical languages

The results proved for the First Hilbert problem by Gödel and Cohen discuss how the Continuum Hypothesis is connected to Zermelo–Fraenkel set theory. Thus, the problem has been considered in two logical steps: (i) efforts have been made to understand what is an accurate definition of the concept 'set'; (ii) the hypothesis was studied using this definition. Such a way of reasoning is quite common in Mathematics. The idea of finding an adequate set of axioms for one or another field of Mathematics is among the most attractive goals for mathematicians. Usually, when it is necessary to define a concept or an object, logicians try to introduce a number of axioms describing the object. However, this way is fraught with danger because of the following reasons.

First, when we describe a mathematical object or concept we are limited by the expressive capacity of the language we use to make this description. A richer language allows us to say more about the object and a weaker language – less. Thus, the incessant development of the mathematical (and not only mathematical) languages leads to the continual necessity of a transcription and the further specification of axiomatic systems. Second, there is no guarantee that the chosen axiomatic system defines 'sufficiently well' the required concept and a continual comparison with practice is required in order to check the goodness of the accepted set of axioms. However, there cannot be again any guarantee that the new version will be the last and definitive one because we do not know which new facts related to the studied object will be discovered in the future. Finally, the third limitation has been discovered by Gödel in his two famous incompleteness theorems (see [4]).

Thus, if we return to the subject of the First Hilbert problem, we should observe that people successfully measured finite sets for centuries without having a 'precise' definition of the notion 'set'. Moreover, the possibility itself of giving such a definition is questionable (in fact, scientific debates in this area persist (see, e.g., [17, 18])). In this paper, we propose a new point of view on the First Hilbert problem by concentrating our attention on various processes of counting and mathematical languages used for this purpose.

We start by noticing that in linguistics the relativity of descriptions of the world around has been formulated in the form of the Sapir-Whorf thesis (see [2, 7, 9, 10, 12]) also known as the 'linguistic relativity thesis'. As becomes clear from its name, the thesis does not accept the idea of the universality of language and challenges the possibility of perfectly representing the world with language, because it implies that the mechanisms of any language condition the thoughts of its speakers.

In this paper, we study the relativity of mathematical languages in situations where they are used to observe and to describe finite and infinite sets. Let us first illustrate the concept of the relativity of mathematical languages by the following example. In his study published in *Science* (see [6]), Peter Gordon describes a primitive tribe living in Amazonia – Pirahã – that uses a very simple numeral system[1] for counting: one, two, 'many'. Sometimes they use the numerals I, II, and III to indicate these numbers. For Pirahã, all quantities larger than two are just 'many' and such operations as 2+2 and 2+1 give the same result, i.e., 'many'. By using their weak numeral system Pirahã are not able to see, for instance, numbers 3, 4, and 5, to execute arithmetical operations with them, and, in general, to say anything about these numbers because in their language there are neither words

[1]We remind that *numeral* is a symbol or group of symbols that represents a *number*. The difference between numerals and numbers is the same as the difference between words and the things they refer to. A *number* is a concept that a *numeral* expresses. The same number can be represented by different numerals. For example, the symbols '10', 'ten', and 'X' are different numerals, but they all represent the same number (see [9] for a detailed discussion).

nor concepts for that. As a consequence, when they observe a set having three elements and another set having 4 elements their answer is that both sets have ‘many’ elements.

It is important to emphasize that their answer ‘many’ is correct in their language and if one is satisfied with its accuracy, it can be used (and *is used* by Pirahã) in practice. Note that also for us, people knowing that $2+1=3$ and $2+2=4$, the result of Pirahã is not wrong, it is just *inaccurate*. Thus, if one needs a more precise result than ‘many’, it is necessary to introduce a more powerful mathematical language (a numeral system in this case) allowing one to express the required answer in a more accurate way. By using modern powerful numeral systems where additional numerals for expressing the numbers ‘three’ and ‘four’ have been introduced, we can notice that within ‘many’ there are several objects and the numbers 3 and 4 are among them. Therefore, these numbers can be used for various purposes, in particular, for counting the number of elements in sets.

Among other things this example shows us that a mathematical language can contain numerals required to formulate a question but not the numerals that can express the corresponding answer with the desired accuracy. Thus, the choice of the mathematical language depends on the practical problem that is to be solved and on the accuracy required for such a solution. In dependence of this accuracy, a numeral system that would be able to express the numbers composing the answer (and the intermediate computations, if any) should be chosen.

Such a situation is typical for natural sciences where it is well known that instruments bound and influence results of observations. When physicists see a black dot in their microscope they cannot say: The object of the observation *is* the black dot. They are obliged to say: the lens used in the microscope allows us to see the black dot and it is not possible to say anything more about the nature of the object of the observation until we change the instrument - the lens or the microscope itself - by a more precise one. Then, probably, when a stronger lens will be put in the microscope, physicists will be able to see that the object that seemed to be one dot consists of two dots.

Note that both results (one dot and two dots) correctly represent the reality with the accuracy of the chosen instrument of the observation (lens). Physicists decide the level of the precision they need and obtain a result depending on the chosen level of the accuracy. In the moment when they put a lens in the microscope, they have decided the minimal (and the maximal) size of objects that they will be able to observe. If they need a more precise or a more rough answer, they change the lens in their microscope. Analogously, when mathematicians have decided which mathematical languages (in particular, which numeral systems) they will use in their work, they have decided which mathematical objects they will be able to observe and to describe.

In natural sciences, there always exists the triad – the researcher, the object of investigation, and tools used to observe the object – and the instrument used to observe the object bounds and influences results of observations. The same happens in Mathematics studying numbers and objects that can be constructed by using numbers. Numeral systems used to express numbers are among instruments of observations used by mathematicians. The usage of powerful numeral systems gives the possibility of obtaining more precise results in Mathematics in the same way as the usage of a good microscope gives the possibility to obtain more precise results in Physics.

Let us now return to Pirahã again. Their numeral system has another interesting feature particularly interesting in the context of the study presented in this paper:

$$\text{‘many’} + 1 = \text{‘many’}, \quad \text{‘many’} + 2 = \text{‘many’}, \quad \text{‘many’} + \text{‘many’} = \text{‘many’}. \quad (1)$$

These relations are very familiar to us in the context of our views on infinity used in the Calculus

$$\infty + 1 = \infty, \qquad \infty + 2 = \infty, \qquad \infty + \infty = \infty. \quad (2)$$

Thus, the modern mathematical numeral systems allow us to distinguish larger (with respect to Pirahã) finite numbers but when we speak about infinite numbers, they give results similar to those obtained by Pirahã. Formulae (1) and (2) lead us to the following observation: *Probably our difficulty in working with infinity is not connected to the nature of infinity but is a result of inadequate numeral systems used to express infinite numbers.*

This remark is important with respect to the First Hilbert problem because its formulation and its investigations executed so far used mathematical instruments developed by Georg Cantor (see [1]) who has shown that there exist infinite sets having different number of elements. In particular, the First Hilbert problem deals with two kinds of infinite sets – countable sets and the continuum. Cantor has proved, by using his famous diagonal argument, that the cardinality, $\aleph_0$, of the set, $\mathbb{N}$, of natural numbers is less than the cardinality, C, of real numbers $x \in [0,1]$.

Cantor has also developed an arithmetic for the infinite cardinal numbers. Some of the operations of this arithmetic including numerals $\aleph_0$ and C are given below:

$$\aleph_0 + 1 = \aleph_0, \qquad \aleph_0 + 2 = \aleph_0, \quad (3)$$

$$C + 1 = C, \qquad C + 2 = C, \qquad C + \aleph_0 = C. \quad (4)$$

Again, it is possible to see a clear similarity with the arithmetic operations used in the numeral system of Pirahã. This similarity becomes even more pronounced when one looks at the numeral system of another Amazonian tribe – Mundurukú – who fail in exact arithmetic with numbers larger than 5 but are able to compare and add large approximate numbers that are far beyond their naming range (see [11]). Particularly, they use the words ‘some, not many’ and ‘many, really many’ to distinguish two types of large numbers. It is sufficient to substitute in (3), (4) these two words with the numerals $\aleph_0$ and C, respectively, to have an idea about their arithmetic.

Thus, the point of view on Mathematics, and, in particular, on numbers and sets of numbers ($\mathbb{N}, \mathbb{Z}, \mathbb{R}$, etc.) used in this paper follows natural sciences and consists of the following. There exist mathematical objects (e.g., numbers and sets of numbers) that are objects of the observation. There exist different numeral systems (instruments of the observation) allowing us to observe certain numbers through numerals belonging to these numeral systems, to execute certain operations with them, and, in particular, to use these numerals for measuring sets of numbers. The researcher (the observer) is able to view only those numbers that are visible through numerals available in one or another numeral system. Sometimes a fixed numeral system, $\mathcal{S}_1$, allows the researcher to obtain a precise answer that cannot be improved by the usage of a more sophisticated numeral system, $\mathcal{S}_2$, and sometimes the introduction of $\mathcal{S}_2$ allows him/her to improve the accuracy of the result.

For example, let us indicate the numeral system of Pirahã as $\mathcal{S}_{\mathcal{P}}$ and the numeral system of Mundurukú as $\mathcal{S}_{\mathcal{M}}$. Then the accuracy of expression of the numbers 1 and 2 by $\mathcal{S}_{\mathcal{P}}$ cannot be improved by switching to the usage of $\mathcal{S}_{\mathcal{M}}$. However, the accuracy of the answer 'many' can be improved by $\mathcal{S}_{\mathcal{M}}$ when numbers less than six are observed. The system $\mathcal{S}_{\mathcal{M}}$, in its turn, starts to give inaccurate answers when one starts to use it to observe numbers larger than five. Modern numeral system (including numerals ∞, $\aleph_0$, $\aleph_1$, C, ϖ, etc.) can improve this accuracy for large finite numbers but start to give inaccurate answers when infinite (and infinitesimal) numbers should be observed.

This clear separation between mathematical objects and mathematical languages used to observe and to describe these objects means, in particular, that from this point of view, *axiomatic systems do not define mathematical objects but just determine formal rules for operating with certain numerals reflecting only some (not all) properties of the studied mathematical objects*. For example, axioms for real numbers are considered together with a particular numeral system $\mathcal{S}$ used to write down numerals and are viewed as practical rules (associative and commutative properties of multiplication and addition, distributive property of multiplication over addition, etc.) describing operations with the numerals. The completeness property is interpreted as a possibility to extend $\mathcal{S}$ with additional symbols (e.g., e, π, $\sqrt{2}$, etc.) taking care of the fact that the results of computations with these symbols agree with the facts observed in practice. As in Physics, it becomes impossible to say anything about the object of the observation without the usage of an instrument of the observation. In case of mathematical objects, the used mathematical languages (including numeral systems) are among these instruments.

Another important consequence of the understanding of the existence of mathematical objects and their separation from tools of the observation consists of the fact that it is necessary to be very careful when one speaks about sets of numbers. In particular, when we speak about sets (finite or infinite) we should take care about tools used to describe a set. In order to introduce a set, it is necessary to have a language (e.g., a numeral system) allowing us to describe its elements and the number of the elements in the set. For instance, the set

$$A = \{1, 2, 3, 4, 5\} \tag{5}$$

cannot be defined using the mathematical language of Pirahã.

Such words as ‘the set of all finite numbers’ do not define a set completely from our point of view, as well. It is always necessary to specify which instruments are used to describe (and to observe) the required set and, as a consequence, to speak about ‘the set of all finite numbers expressible in a fixed numeral system’. For instance, for Pirahã ‘the set of all finite numbers’ is the set $\{1,2\}$ and for Mundurukú ‘the set of all finite numbers’ is the set A from (5). In modern numeral systems there exist similar limitations and hence our possibilities to write down numbers are limited (e.g., it is not possible to write down in a positional numeral system any number that has 10^{100} digits)[2]. We emphasize again that, as it happens in Physics, the instrument used for an observation bounds the possibility of the observation. It is not possible to say how we shall see an object of our observation if we have previously not clarified which instruments are to be used to execute the observation.

Thus, we have emphasized in this section that there exists the crucial difference between numbers and numeral systems used to observe them and, analogously, between sets of numbers and sets of numerals used to observe them. Advanced contemporary numeral systems enable us to distinguish within ‘many’ various large finite numbers. As a result, we are able to use large finite numbers in computations, to construct mathematical models involving them, and, in particular, to measure large finite sets. Analogously, it can be expected that if we would be able to distinguish more infinite numbers we could execute computations with them and measure infinite sets.

3 Mathematical objects discussed in the First Hilbert problem and the accuracy of mathematical languages used for their description

Let us start by making some observations with respect to the formulation of the part (i) of the First Hilbert problem (see Introduction). It uses the mathematical language developed by Cantor and considers real numbers, natural numbers, cardinal numbers, countable sets, and the continuum. In the formulation there is no any distinction between objects of the observation (infinite sets and numbers) and the mathematical language used for the observation (in particular, the words ‘countable sets’ and ‘continuum’). Let us reconsider mathematical objects and numeral systems involved in the formulation of the problem by concentrating our attention on this distinction.

The real numbers are understood in the formulation (see page 2) to be ‘the points of a line’. Then, the existence of the continuum has been demonstrated by Cantor using the diagonal argument working with a positional representation of

[2] It is interesting that this problem was clear to Archimedes. In his work *The Sand Reckoner*, in order to express huge numbers he had invented new numerals. It was necessary because the ancient Greeks used a simple system for writing numbers using 27 different letters of the Greek alphabet. This system did not allow Archimedes to write down huge numbers he wished to work with.

points

$$(.a_1a_2a_3\dots a_{i-1}a_ia_{i+1}\dots)_b \qquad (6)$$

in the unitary interval where the symbol b indicates a finite radix of the positional system and symbols $a_i, 0 \le a_i \le b-1$, are called digits. Thus, the object of the observation – the set of real numbers over the unitary interval – is observed through the positional numeral system (6) with the radix b.

Then the numeral system of infinite cardinals is used to observe infinite numbers and to measure infinite sets. In particular, two cardinal infinite numbers expressed by the numerals $\aleph_0$ and C are used to measure the infinite sets of natural and real numbers in the formulation of the First Hilbert problem.

Let us make some important observations. First, Cantor has proved, that the number of elements of the set $\mathbb{N}$ is smaller than the number of real numbers at the interval $[0,1)$ *without calculating the latter*. Second, he worked with the numerals (6) supposing that they represent *all real numbers* over the unitary interval. Third, he has also proved that the introduced numeral C is the cardinality also of the whole real line.

However, as it has been discussed in the previous section, the arithmetic using numerals $\aleph_0$ and C clearly indicates that the accuracy of this numeral system is not satisfactory when it is necessary to measure infinite sets (e.g., addition of an element to any infinite set cannot be registered, see (3), (4)). This numeral system allows us to distinguish sets having 'some, not many' elements from the sets having 'many, really many' elements but does not give any possibility to distinguish the measure of sets D and $\tilde{D}$ if both D and $\tilde{D}$ have 'some, not many' elements or both D and $\tilde{D}$ have 'many, really many' elements.

Let us consider, for example, the set of integers, $\mathbb{Z}$, and the set of natural numbers, $\mathbb{N}$. The result saying that both of them have the same cardinal number $\aleph_0$ can be viewed as an indication of the fact that the numeral system of cardinals is too weak to discern the difference of the number of elements of these two sets. An analogous comment can be made with respect to the fact that C is the cardinality of both the set of all real numbers, $\mathbb{R}$, and the set of real numbers over the unitary interval being a proper subset of $\mathbb{R}$.

Thus, a more powerful numeral system that would allow us to capture these differences is required to measure infinite sets. Recently, a new numeral system has been developed to express finite, infinite, and infinitesimal numbers in a unique framework (a rather comprehensive description of the new methodology can be found in the survey [15]). The main idea consists of measuring infinite and infinitesimal quantities by different (infinite, finite, and infinitesimal) units of measure using a new numeral.

A new infinite unit of measure has been introduced for this purpose as the number of elements of the set $\mathbb{N}$ of natural numbers. It is expressed by a new numeral ① called *grossone*. The new approach does not contradict Cantor and can be viewed as an evolution of his deep ideas regarding the existence of different infinite numbers in a more applied and precise way.

In the new numeral system using ①, it becomes possible to express a variety of numerals that allow one to observe infinite positive integers and to use them for measuring certain infinite sets. As a consequence, the new numeral system allows us to improve the accuracy of measuring infinite sets in comparison with the numeral system using cardinals introduced by Cantor. For instance, within the countable sets it becomes possible to distinguish the following sets having the different number of elements: the set of even numbers has $\frac{①}{2}$ elements; the set of integers, $\mathbb{Z}$, has 2①+1 elements; the set $\mathbb{N}_- = \mathbb{N} \setminus \{x\}$, $x \in \mathbb{N}$, has $① - 1$ elements; the set $\mathbb{N}_+ = \mathbb{N} \bigcup \{y\}$, $y \notin \mathbb{N}$, has $① + 1$ elements; and the set

$$P = \{(a_1, a_2) : a_1 \in \mathbb{N},\ a_2 \in \mathbb{N}\}$$

has $①^2$ elements. Another example of the usage of the new numerals can be given for the set $B = \{3,4,5,69\} \cup (B_1 \cap B_2)$, where $B_1 \subset \mathbb{N}$, $B_2 \subset \mathbb{N}$ and

$$B_1 = \{4, 9, 14, 19, 24, \ldots\}, \qquad B_2 = \{3, 14, 25, 36, 47, \ldots\}.$$

As it has been shown in [15], the set B has $\frac{①}{55} + 3$ elements.

With respect to the sets that were indicated by Cantor as having cardinality of the continuum the new approach also allows one to obtain more precise results and to notice that among sets having cardinality C there exist sets having the different infinite number of elements. First of all, it can be shown (see [15]) that numerals (6) do not represent *all* real numbers within the unitary interval. They represent only those numbers that can be expressed by the numerals (6) that have no more than grossone digits. For instance, the irrational number $\frac{\sqrt{e}}{4}$ cannot be expressed in it because even grossone digits are not sufficient to express e in the positional numeral system (see [15]).

Moreover, the new approach allows us to show that the numbers, n_b, of all numerals expressible in the form (6) are different depending on the radix b and to calculate that $n_b = b^{①}$. For instance, the binary numeral system ($b = 2$) contains $n_2 = 2^{①}$ numerals. This is smaller than, for example, the number of numerals expressible in the decimal positional system having $n_{10} = 10^{①}$ numerals. As a result, the binary numeral system can express less numbers than the decimal one. It is important to emphasize that for any finite $b > 1$ all of the obtained numbers, $b^{①}$, are larger than ① being the number of elements of the set, $\mathbb{N}$, of natural numbers. This fact can be viewed as a further specification of the results obtained by Cantor.

We can also consider positional numeral systems having an infinite number of positions k in (6) such that $k < ①$. Then these sets will have less than $b^{①}$ numerals. For instance, for $k = ①/2$, we have the following numeral system

$$(.a_1 a_2 a_3 \ldots a_{0.5①-1} a_{0.5①})_b$$

that contains $b^{①/2}$ numerals.

Another important particular case can be obtained from the solution to the equation $b^x = ①$ that tells us that the positional numeral systems having $k_1 =$

$[\log_b(①)]$ and $k_2 = [\log_b(①)] + 1$ positions in their numerals can express $y_1 \leq ①$, and $y_2 > ①$ numbers, respectively, where $[u]$ is the integer part of u. The positional numeral system with the numerals having k_1 digits is of special interest in the context of the Continuum Hypothesis and we call it *critical value*. This terminology is used because even though in this case we deal with a positional system with numerals consisting of an infinite number of digits, the quantity, y_1, of numerals expressible in this numeral system is less than the number of elements of the set of natural numbers. By a complete analogy critical values can be calculated not only with respect to grossone but for other infinite integers, as well. For instance, this can be done for $①/2$ (the number of elements of the set of even numbers), for $2① + 1$ (the number of elements of the set of integers), etc.

Let us notice now that other numeral systems can be used to express points of an interval (see [15]) and that the new methodology allows us to introduce a more physical concept of continuity (see [16]). Recall that in Physics, the 'continuity' of an object is relative. For example, if we observe a table by eye, then we see it as being continuous. If we use a microscope for our observation, we see that the table is discrete. This means that *we decide* how to see the object, as a continuous or as a discrete, by the choice of the instrument for the observation. A weak instrument – our eyes – is not able to distinguish its internal small separate parts (e.g., molecules) and we see the table as a continuous object. A sufficiently strong microscope allows us to see the separate parts and the table becomes discrete but each small part now is viewed as continuous.

In contrast, in traditional mathematics, any mathematical object is either continuous or discrete. For example, the same function cannot be both continuous and discrete. Thus, this contraposition of discrete and continuous in the traditional mathematics does not reflect properly the physical situation that we observe in practice. The numeral system including grossone gives us the possibility to develop a new theory of continuity that is closer to the physical world and better reflects the new discoveries made by the physicists (see [16]).

Let us pass now from the unitary interval to the whole real line. To observe real numbers (the object of the observation) it is necessary first to choose a numeral system (the instrument of the observation) to represent them. After this choice has been done, it becomes possible to count the number of numerals expressible in the fixed numeral system and to understand how many real numbers can be expressed in this numeral system. For instance, let us consider numerals

$$\pm(\ldots a_2 a_1 a_0 . a_1 a_2 a_3 \ldots)_b \tag{7}$$

that are expressed in the positional system with the radix b. Their number is equal to $2b^{2①}$ if we want to use in (7) sequences of integer and fractional digits consisting of ① elements each (see [15]). Note that $2b^{2①}$ is the number of different numerals. In this numeral system the number zero can be represented by two different numerals

$$-\underbrace{000\ldots000}_{①\ \text{digits}}.\underbrace{000\ldots000}_{①\ \text{digits}}, \qquad +\underbrace{000\ldots000}_{①\ \text{digits}}.\underbrace{000\ldots000}_{①\ \text{digits}}.$$

Suppose not that we want to change the numeral system for representation of real numbers and decide to work, e.g., with the numerals

$$\pm(.a_1a_2a_3\ldots)_b \cdot b^{\pm(p_1p_2p_3\ldots)_b} \qquad (8)$$

where digits $0 \leq a_i < b$ and $0 \leq p_i < b$, and both sequences $\{a_i\}$ and $\{p_i\}$ consist of ① elements each. Then it is easy to show that this numeral system contains $4b^{2①}$ different numerals.

We emphasize again that any numeral system allows us to express precisely the number of elements only of certain sets. For the set (5) the numeral system, $\mathcal{S}_{\mathcal{M}}$, of Mundurukú allows us to do this. For the sets $\mathbb{N}$ and $\mathbb{Z}$ the numeral system including grossone is sufficient. To express the number of elements of the set of real numbers even this numeral system is not sufficient. Even though it allows us to specify certain infinite subsets of the set of real numbers and to count their number of elements, it is too weak for both expressing the number of elements of $\mathbb{R}$ and expressing irrational numbers. Note that with respect to the latter it behaves as all other existing numeral systems which use one of the following two alternatives: (a) use approximations of irrational numbers expressible in the chosen numeral system; (b) use numerals created ad hoc for the required irrational numbers, e.g. $e, \pi, \sqrt{2}, \sqrt[5]{3}$, etc.

Let us consider now the second part of the First Hilbert Problem and give some comments upon. Again, from our point of view, its difficulty consists of the fact that in its formulation there is no distinction between the object of the observation and the numeral systems used for the observation.

In our analysis, we first emphasize that in order to compare two real numbers it is necessary to have a numeral system allowing us to express both of them. Then again, as it was in Physics with the choice of the lens for the microscope, we have decided which numbers can be compared at the moment when we have chosen our numeral system. This is very important from the methodological point of view: we are not able to compare numbers if we have no numerals allowing us to express these numbers (at least in an approximative way).

For instance, let us consider the set, $\overline{\mathbb{N}}$, of all natural numbers that we are able to write down using all known numeral systems. Then, we are not able to write down, to compare, and to order numbers belonging to the set $\mathbb{N} \setminus \overline{\mathbb{N}}$. This set can be defined but we are not able to indicate any of its elements, including the first one. Thus, it is possible to speak about the existence of this set but it is impossible to execute operations with its elements because we have no tools to express them.

Note also the importance of the fact that if we want to compare two numbers accurately, they should be expressible in the same numeral system. If they are expressed by two different numeral systems then the problem of a transcription from one to another system arises and it is possible that such a precise transcription is impossible (e.g., it is shown in [15] that e cannot be written in any positional system with a finite radix b even if one has an infinite number of digits in this positional record). In such cases only an approximate comparison can be done ($e \approx 2.7$).

Let us now return to the formulation of the problem where Hilbert writes ‘On the other hand the system of all real numbers, i.e., the continuum in its natural order, is evidently not well ordered. For, if we think of the points of a segment of a straight line, with its initial point excluded, as our partial assemblage, it will have no first element.’ In these phrases Hilbert speaks about numbers expressible by numerals (6) as about all real numbers. This is not the case as it has been shown in the analysis above. Several numeral systems that can be used to express real numbers have been studied in [15] in detail.

With respect to the second phrase of Hilbert it is possible to make the following comment. Without loss of generality, we consider the unitary interval as a segment of a straight line and we use numerals (6) to express numbers Hilbert speaks about. It has been shown in [15] for these numerals (other numeral systems have been studied in [15], as well) that all of them represent different numbers. Thus, if we have decided to observe the interval $[0,1]$ by numerals (6) with $b = 10$ then only the following $10^{①}$ numbers can be distinguished within this interval

$$0.\underbrace{000\ldots000}_{①\text{ digits}},\ 0.\underbrace{000\ldots0001}_{①\text{ digits}},\quad \ldots\quad 0.\underbrace{999\ldots998}_{①\text{ digits}},\ 0.\underbrace{999\ldots999}_{①\text{ digits}}.$$

Then, if we decide to exclude from the interval $[0,1]$ the number zero represented by the numeral $0.\underbrace{000\ldots000}_{①\text{ digits}}$ then the next numeral, $0.\underbrace{000\ldots0001}_{①\text{ digits}}$, gives us the first element in the set of remaining numerals (that has now $b^{①}-1$ elements). Obviously, if the accuracy of the numeral system (6) is not sufficient for a problem we want to solve over the interval $[0,1]$, then we can decide to add some new numerals to deal with. For instance, we can decide to add the numeral $1.\underbrace{000\ldots000}_{①\text{ digits}}$ in order to represent the number one exactly. Then, this numeral and numerals (6) give us the possibility to distinguish within the interval $b^{①}+1$ different numbers.

Therefore, the numbers that can be expressed by the numerals (6) and that we are able to write down can be ordered. This note is important because we are not able to write down all numerals of the type (6), e.g., the ones having an infinite number of different digits. However, the first elements Hilbert speaks about in the text quoted above can be expressed and ordered. Without loss of generality we give an example using the decimal positional system and write down the first four numerals from this numeral system:

$$0.\underbrace{000\ldots000}_{①\text{ digits}} < 0.\underbrace{000\ldots0001}_{①\text{ digits}} < 0.\underbrace{000\ldots0002}_{①\text{ digits}} < 0.\underbrace{000\ldots0003}_{①\text{ digits}}.$$

By a complete analogy, the smallest numbers expressible in the numeral systems (7) and (8) can be indicated.

4 Conclusion

Two new mathematical instruments have been applied in the paper to study the First Hilbert problem: the methodology distinguishing mathematical objects and mathematical languages used to describe these objects; and the new numeral system allowing one to express different infinite numbers and to use these numbers for measuring infinite sets.

With respect to the Continuum Hypothesis it has been established that the mathematical language used by Cantor and Hilbert to formulate the problem has two peculiarities. First, it does not take into consideration the difference between mathematical objects under the observation and the mathematical language used to describe these objects. Second, this language does not allow one to discern among different numerable sets those sets that have the different infinite number of elements. The same happens with respect to different sets having the cardinality of the continuum.

The new numeral system has allowed us to measure certain infinite sets with a high accuracy and to express explicitly the number of their elements. In particular, the sets mentioned in the Continuum Hypothesis have been studied in detail and a constructive answer to the Hypothesis has been given in the framework of the new methodology.

It is obligatory to emphasize that both approaches, that of Cantor and Hilbert and the new one, do not contradict one another. Both of them represent the reality but they do it with the different accuracy. Such a situation is typical for natural sciences where it is well known that instruments bound and influence results of observations. As a result, the moment when a researcher chooses an instrument for the required observation is the moment when the accuracy of the observation is determined. We can illustrate this situation by using again the analogy with the microscope.

Suppose that with a weak lens a physicist sees two dots in the microscope and with a stronger lens instead of the two dots he/she sees 45 smaller dots. Both answers are correct but they describe the object of the observation with the different accuracy that is determined by the instrument (the lens) chosen for the observation. Analogously, when mathematicians have decided which mathematical languages (in particular, which numeral systems) they would use in their work, they have decided which mathematical objects they would be able to observe and to describe.

The language used by Cantor and Hilbert allows an observer to see in the mathematical microscope two dots (numerable sets and the continuum) and the accuracy of the used lens (the numeral system using cardinals) does not allow the observer to capture the presence of eventual other dots. In addition, the observer does not realize that he/she has put on the sample stage of the microscope instead of the entire object (real numbers) just a part of it (namely, numbers expressible in a positional numeral system with the infinite number of digits). Moreover, the observer is not conscious that even the taken part is not always the same because it is changed when the radix of the positional numeral system is changed.

In the case of the new mathematical language, the observer distinguishes the lens from the objects of the observation (numeral systems used to describe sets from the sets themselves) and has a stronger lens (the numeral system using grossone) allowing him/her to observe instead of two dots many different dots (various infinite sets having the different number of elements). In particular, it becomes possible to distinguish various sets of numerals expressible by positional systems with different radixes and different infinite lengths of digits. It has been shown that for a fixed radix a critical length can been established such that sets of numerals with a length superior to the critical one have more elements than the set of natural numbers. For infinite lengths inferior to the critical value the number of numerals expressible by this positional system is inferior to the number of elements of the set of natural numbers. Analogous critical values can be established for other numerable sets.

With respect to the second part of the problem dealing with well ordered sets it has been commented that it is necessary to operate very carefully with propositions dealing with the existence of non trivial mathematical objects and with executing operations with them. In particular, if such a proposition supposes the execution of an operation, then it is necessary to verify the existence of mathematical tools (e.g., numeral systems) allowing us to express mathematical objects required for this operation including both the operands and the intermediate and final results.

It has been emphasized that we are not able to compare numbers if we have no numerals allowing us to express these numbers (at least in an approximative way). As a consequence, only those numbers can be ordered for which numeral systems allowing one to express them are known. Finally, it has been shown that minimal numbers expressible in positional numeral systems with the different infinite quantity of digits can be ordered.

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