A grossone-based numerical model for computations with infinity: a case study in an Italian high school

Francesco Ingarozza[0000-0002-8378-399X], Maria Teresa Adamo, Maria Martino, and Aldo Piscitelli

Liceo Scientifico Statale "Filolao",
via Acquabona, Crotone (KR) 88821, Italy
francesco81tfa@gmail.com, mt.adamo@hotmail.it,
mariafilomena.martino@gmail.com, aldo.piscitelli@inwind.it

Abstract. The knowledge and understanding of abstract concepts systematically occur in the studies of mathematics. The epistemological approach of these concepts gradually becomes of higher importance as the level of abstraction and the risk of developing a "primitive concept" which is different from the knowledge of the topic itself increase. A typical case relates to the concepts of infinity and infinitesimal. The basic idea is to overturn the normal "concept-model" approach: no longer a concept which has to be studied and modeled in a further moment but rather a model that can be manipulated (from the calculation point of view) and that has to be associated to a concept that is compatible with the calculus properties of the selected model. In this paper the authors want to prove the usefulness of this new approach in the study of infinite quantities and of the infinitesimal calculus. To do this, they expose results of an experiment being a test proposed to students of a high school. The aim of the test is to demonstrate that this new solution could be useful in order to enforce ideas and acknowledgment about infinitesimal calculus. In order to do that, the authors propose a test to their students a first time without giving any theoretical information but only using an arithmetic/algebraic model. In a second moment, after some lectures, the students repeat the test showing that new better results come out. The reason is that after lessons, students could join new basic ideas or primitive concepts to their calculus abilities. By such doing they do not use a traditional "concept–model" but a new "model–concept" solution.

Keywords: Mathematics education · Teaching/learning methods and strategies · Grossone · Computer tools.

1 Introduction

The understanding and knowledge of abstract concepts or of infrequent concepts, are that of an element which systematically occurs in the study of mathematics (see [16, 23]). Therefore the epistemological approach in regards to these concepts, becomes fundamental to give guarantees of success in terms of didactic results. The sense of this observation is very important. Besides it increases as the same level of abstraction of mathematics concepts and related models increases. The reason for that sentence is that as the level of abstraction increases and becomes considerable, it increases the risk to develop those fundamental ideas which could be different form the real being of knowledge. Those fundamental ideas are defined primitive concepts; they could be very far from the real subject of knowledge. Not only, that ideas could be misleading with the respect to the whole ideas of knowledge (see [3, 5, 13–15, 17, 18]). One of the abstraction elements that high school students encounter is related to the concepts of infinity and infinitesimal (see, for example, [12, 16, 22, 24, 27, 28]). Therefore it becomes mandatory a path of knowledge, of discussion about traditional teaching. Such a strategy can be one, which makes use of a symbolic association of a "symbol-concept" model. This way it becomes possible to normalize a process based on an abstract concept, whose in-depth discussion requires a convergence of different signifiers.

This normalization process can be reached through the assumption of a single shared model, such a syllabus based on an epistemological model (see [19, 21]). The basic idea is to overturn the normal "concept-model" approach: no longer a concept to be studied and modelled (only in that of a second time), but a model that can be manipulated (from the point of view of calculation) and to which it is possible to associate a concept. The associated (to the model) concept is compatible with the calculation properties of the identified model and it's also compatible with the theoretical results which can be presented (see, for instance, [3, 5, 21]). The described approach is the one which has been used in this didactic experiment. It represents a new idea for the presentation of the concepts of infinity and infinitesimal. The objective is therefore the ability to express a concept regardless of the intrinsic difficulties of the same representation. For the infinite quantity, the symbol ① (*"grossone"*) is used. By using this model-based approach, the "symbol-concept" association overcomes the operational limits of a no adequate language. In the following discussion we present the results of a didactic experimentation. This experiment is aimed at demonstrating the validity and effectiveness, in terms of didactic repercussions. The demonstration is based on a real didactic case which is the study of infinitesimal calculus. Through this case we want to prove the validity and effectiveness of the "symbol-concept" approach and of the model based approach in computational arithmetic (see [4, 12, 19, 21, 23, 25]). It is important to note that another similar experiment has been carried out independently during the same periods (see [1]).

It is important to remark that the content of this paper concerns about the Arithmetic of infinite (based on the use of the symbol ①). This idea was developed starting from 2003 by Yaroslav Sergeyev (see, e.g., [22, 23, 25, 26]). We should notice that there are different examples in the scientific literature that can confirm the usefulness of such an approach in different fields of mathematics (see [2, 6–11, 19, 20, 22–27]). Infinite Arithmetic represents a new and different point of view on the concept of number and its related concept of counting. Starting from their own experiences, the authors can confirm the difficulties of the traditional calculus that students face, especially when they deal with counting. For example, difficulties related to the study of probabilities are originated by the difficulties of counting events. The new approach can be applied both to already known problems and to new ones. For the first class of problems this method can confirm well known results while for new ones it can represent an alternative way for finding solutions. Regarding the study of infinitesimal calculus, the grossone based model can represent a good solution which can be easily understood by high school students.

2 Description of the experimentation

In the following two subsections we deal with the sample and the pursued objectives and purposes.

2.1 The sample

The didactic experiment was directed to the students in their 4th year at the secondary school *Liceo Scientifico Filolao*, in Crotone, Italy. In particular, the sample was evaluated by three classes (from now we call them class A, class B and class C in order to save the privacy of the students) and involved a total of about 70 students; it lasted about 8 hours and was carried out in the period of April/May 2019. The choice of the sample and of the period is not random but it has a precise reason: during this period those students started to approach infinitesimal calculus and exactly limits operations. Because of these considerations, it is correct to suppose that there are at least four reasons to consider the chosen sample and the period to be good choices:

- Confidence with the concepts exposed in the experiment;
- Students know the difficulties in regards to a symbolic-arithmetic manipulation of these subjects;
- The possibility to verify, after the experiment, that it is possible to have an algebraic and computational manipulation of the topics;
- The possibility to verify the advantages deriving from this model-based approach for infinitesimal calculus.

It should be reminded that all the classes involved in the experiment have, as planned, already operated with elements of infinitesimal calculus and with the study of indeterminate forms.

2.2 Objectives and purposes

The experiment carried out on the selected sample was not only a didactic experiment but also an epistemological experiment: the target of the experiment was twofolded. On the one hand, the well-known importance of presenting concepts that require a good deal of abstraction as clearly as possible and far from ambiguities, doubts and perplexities, and jointly, a good mastery of arithmetical, algebraic and computational skills with regard to presented concepts. Near to this aspect, purely didactic, from an epistemological point of view, it was/is important to verify the didactic efficacy of the approach used, starting from the verification and clear comprehension, and with the mastery of the calculation properties of the concepts presented to be able to draw appropriate considerations about the usefulness of the presented approach (see [13, 15, 18]).

3 Results of the experiment

The sample was submitted to a didactic experiment which was made up by three distinct phases:

- Test administration;
- Didactic lectures oriented on commenting the test;
- Test administration.

The basic idea of the experiment is to highlight some shadow areas which are related to some aspects inherited by infinitesimal calculus and especially to the formal treatment of the concept of infinity. In order to demonstrate the didactic efficiency of the model based approach, a preliminary test was submitted to the students; the aim of this test was to verify good computational capacities linked to some symbols in the face of a limited understanding of some concepts or of some obvious gaps in the complete and full understanding of those same concepts. In particular, the grossone symbol was presented within the preliminary test but a complete description of grossone and its properties was not given. Only one information was provided to the students, in the introduction of the preliminary test. The information is the clarification of the property $① > n, \ \forall$ finite $n \in \mathbb{N}$ (for obvious reasons of formal completeness, see also [19]). After doing that clarification, the students were given a test of 46 questions divided into four macro areas:

- Section 1 - Order relationships;
- Section 2 - Arithmetic;
- Section 3 - Equality;
- Section 4 - Counting of the elements of known numerical sets.

In Section 1 there are questions underlying the order relationships constructed starting from the symbol ①. This section presents 5 open questions. It is therefore possible to answer each question by selecting one of the 5 proposed solutions. With regard to the questions contained in Section 1 it is reasonable to expect sufficiently high results. Moreover, the almost non-existent knowledge about ① is not a limitation for a successful result in Section 1. Indeed, the starting hypothesis ($① > n, \ \forall$ finite $n \in \mathbb{N}$) is more than sufficient to answer with a certain degree of accuracy and with a reduced limit of error, to the questions contained in Section 1 concerning trivial order relationships between arbitrary quantities containing ①. In Section 2, six algebraic questions/exercises were submitted. In particular, to pupils were given some expressions to solve. The expressions contain within them the symbol ① (which must be treated as a trivial and simple algebraic quantity, starting from the previous consideration ①$> n$, $\forall n \in \mathbb{N}$). It is evident that since these are exercises of a purely algebraic nature (expressions!) the submission method of Section 2 is that of closed questions: the students have to perform the required calculations and enter the results. We can confirm the same observations made for Section 1 even for Section 2; in this section of a pure arithmetic/algebraic exercises are submitted. More precisely they (the exercises) are expressions containing the quantity ①. They can be treated as expressions of literal calculation that contain precisely the quantity ① instead of a monomial with any literal part (no clarifications or further properties are required on the numerical set which contains the quantity ①).

It is reasonable to expect quite high results for the entire Section 2 and it is quite obvious to suppose that different results (far from expected ones) may be due to trivial calculation errors. In Section 3, questions were proposed in the form of open ended questions, and they are related to the "parity" of quantities obtained starting from ①. Finally, in Section 4, students have to answer to questions regarding the counting of elements of known numerical sets or parts of them. In Section 3 and Section 4 results change considerably. In fact, the only given information about ① is not more sufficient to provide exhaustive clarifications that allow unequivocal and correct answers to the questions which are proposed in Section 3 and in Section 4 (apart from rarely cases such as the first 7/8 questions of Section 4 in which it is possible to answer by using the normal notions of infinitesimal calculation and limit operation). Besides it is important to remember that the test was made in such a way to avoid the incidence on the final result of randomly right answers (especially in sections in which open ended questions are submitted). In order to avoid false positive cases (randomly correct answers) which could distort the evaluation on the didactic impact of the experimentally presented model, it was decided to assign a highly penalizing negative score for each wrong answer. In particular, wrong answers generate a score of -1, which therefore cancels the score generated by a right answer. In this way it is reasonable to expect that the students won't answer the questions which are related to unknown subjects and topics. By using this strategy is possible to suppose a strong reduction of false positive cases; in this way it is possible to immunize the experiment against this potential risk that could affect its actual validity.

Starting from the previous considerations regarding Section 3 and Section 4, including the strategy of answer evaluation, it is therefore reasonable to expect a significant lowering of the results obtained by the students in Section 3 and Section 4. Notice also that in a total of 46 total applications, Section 4 evaluates more than 54% and Section 3 evaluates more than 20% of the total; for this reason the two sections jointly account for over 75% and therefore it is reasonable to expect very low results from the preliminary test. The fundamental idea of the experiment is just this: to verify a radical change of the results of the same test proposed during the initial and final phases, on the same sample. As was already said, there is only one significant difference between these two phases of the test: the test was submitted for the second time to the same sample of students only after they had followed a cycle of lectures regarding the Grossone theory and model. In such way the students could develop a new basic idea and a new way to approach infinitesimal calculus.

The aim of the experiment in its whole is to demonstrate that this new approach could improve the development of new primitive concepts for students. The first aim of the lesson is to speak to the students about the model used for the representation of infinite quantities, that is, the symbol ① which starts as a numerical approach and extends to all its properties (see [19, 23, 25]). The target of the test is precisely to demonstrate that starting form an easy algebraic manipulation of a symbol, an association with more abstract concepts can be useful in order to understand aspects related to the concept. In particular, and in detail (Section 3 and Section 4) a few expedients are sufficient to start from the innovation associated with a mere and new definition. This definition is that of a new numerical set, made up of an infinite number of elements some of which are infinite elements related to infinite quantities. Thanks to this new and different approach, (grossone model based) students could be able to reach a very high level of knowledge, in a new way which could be a logical unexceptionable manner. This is probably the only way, or one of the few available solutions in order to reach this knowledge, in regard to these subjects. The same concepts could be precluded or not easily deducible by using a different strategy so far from that described on this paper, for example, the traditional way (as used and intended in Italian schools).

The last consideration can be confirmed by results of the test which had been submitted for the first time to students. In fact, these students were in the same conditions as the most of students of Italian high schools. They had never used the new approach which is proposed in this article. Such knowledge allows, in the final phase of the experimentation, to be able to carry out the same preliminary test (even if it's called final test) from which better results come out. It can be said that this experiment is an evaluation and measurement model for the behavior of didactic efficiency of the model based approach.

In the following, some examples of questions of the test are reported.

For Section 1:

Let us consider ① symbol as a positive quantity such that ①$> n$ ($\forall$ finite $n \in \mathbb{N}$).

Choose the right order relationship among the different proposed solutions

1) ① and -①
   A) ①> -①
   B) We can't establish an order relationship
   C) -①≤ ①
   D) -①= ①
   E) ①< -①

2) 2① and 7①

A) 2①> 7①
B) 7①= 2①
C) We can't establish an order relationship
D) 2①≥ 7①
E) 2①< 7①

3) (1/2) ① and (2/3)①
A) (1/2)①> (2/3)①
B) (1/2)①= (2/3)①
C) (2/3)①> (1/2)①
D) We can't establish an order relationship
E) -(2/3)①> (1/2)①

For Section 2:

Solve the following exercises:

1) $6(① - 3) - 8(9 + 3①)$
2) $①(3 + ①) - 4①^2 - 3①(1 - ①)$
3) $2[①(3① - 5) + 3(① + 3) - ①(2① + 4① - 11)]$
4) $\frac{1}{4}(① - 2)(① + 2) - \left(\frac{1}{2}① - 1\right)^2$
5) $(①^2 + ① + 1)^2 - (①^2 + ①)^2$
6) $(①^2 + ① + 1)(① - 1) - (① + 1)(①^2 - ① + 1) + 2$

For Section 3:

Determine whether the following quantities are even or odd numbers, by indicating the letter E (even) or O (odd).

1) $2①$
2) $5①$
3) $7① + 1$
4) $2① - 3$
5) $\frac{1}{2}①$
6) $\frac{1}{5}① + 3$
7) $\frac{3}{7}①$
8) $\frac{5}{4}① - 2$
9) $\frac{1}{2}① - 3$
10) $①$

For Section 4 (remind that in this Section 25 questions have been asked):

1) Indicate the correct value of the sum $\infty + 2$, justifying your answer
2) Indicate the correct value of the sum $\infty + \infty$, justifying your answer
3) Indicate the correct value of the product $2 \cdot \infty$, justifying your answer
4) Indicate the correct value of the product $\infty \cdot \infty$, justifying your answer
5) Indicate the correct value of the sum $\infty - \infty$, justifying your answer
...

9) Determine the number of elements of the set $\mathbb{N}$
10) Determine the number of elements of the set $\mathbb{Z}$
11) Let us consider E as the set of even numbers. Determine if the number of elements of the set $\mathbb{N} \setminus P$ is even or odd.
12) Let us consider O as the set of odd numbers. Determine if the number of elements of the set $\mathbb{Z} \setminus O$ is even or odd.
13) Determine the nature (even or odd) of elements of the set $\mathbb{Z} \setminus \{0\}$
14) Determine the nature (even or odd) of elements of the set made up by the first 100 natural numbers minus the first 30 odd numbers

It should be noticed that the whole test is made up of 46 questions: 5 questions for Section 1 (order relationships), 6 questions, more exactly, 6 exercises (or expressions) for Section 2, 10 questions for Section 3 (parity problems) and 25 questions for Section 4. It is important to note that Section 3 and Section 4 are strictly related each to other. In Section 3 students have to find the nature (even or odd) of some quantities made up by the symbol ①. In Section 4 they are called to determine the nature of the number of elements of sets which have infinite elements. How can they do this? How can they count an infinite number of elements? The only way is to use the ① based model as described in the following section.

Another important thing to be noted in Section 4 is the following: students should be able to answer very easily to some questions by studying infinitesimal calculus according to the traditional approach which is used in Italian school. On the other hand, they are not prepared to overcame the second part of Section 4. According to the consideration which has been just done, it is possible to use the results of Section 4 to have an idea of the behavior of the new didactic approach for the study of infinitesimal calculus.

4 The lessons

Lessons took place in the form of frontal teaching. They were characterized by the use of methodologies based on the model of social collaboration and by using a metacognitive approach. Due to these reasons, the model of the Socratic lesson was used. By doing this choice students were led by the teacher to build their knowledge of concepts step by step. The starting point was the commentary of the answers given (and above all not given) during the preliminary test. This comment has revealed two facts: few difficulties in performing simple complex arithmetic operations and at the same time a very big difficulty in counting (this problem is often ignored in the literature, see [3, 13, 18]). This finding was derived from questions such as:

– Is the number of elements of the set $\mathbb{N}$ even or odd?
– How many are the elements of the set $\mathbb{Z}$?

To these questions we tried to give an answer by using a logically unexceptionable path: it is possible not to know if the number of elements of the set $\mathbb{N}$ is even or odd but surely it will be the sum of two numbers which constitute a partition of $\mathbb{N}$, i.e. even numbers and odd numbers. These two numerical sets have the same number of elements and so, it is possible to deduce that whatever the number of these elements (let's call α this number), the number of the elements of $\mathbb{N}$ will always be double (and so 2α). Thus, it is easy to understand that the number of elements of $\mathbb{N}$ is even.

From this consideration we then moved to the definition of ①: the scale value at which all the elements of $\mathbb{N}$ are counted. From this point on the road has been downhill: the set $\mathbb{N}$ of natural numbers having elements from 1 to ① is defined. This set contains infinitely many elements (precisely, ① elements), some of them are finite, the others are infinite (see, e.g., [19, 23, 25, 26]). Then, the set $\widehat{\mathbb{N}}$ is defined as the set which contains positive integers larger than ①, i.e., $\mathbb{N} \subset \widehat{\mathbb{N}}$. From there it was possible with a few tricks that referred to the properties of the rest classes, to go on to discuss all the aspects which allow the students to answer to the most of the question of the test, especially to those of Section 3 and Section 4.

5 Analysis of the results

In this section we show that the experimentation led to results which are broadly in line with the forecasts. The preliminary test has scores which never reach the level of sufficiency (related as the half of the maximum value, equal to 46 points). It is important to note that the most of scores levels are widely achieved in sections 1 and 2 concerning traditional arithmetic. As it it was expected, it can be seen from Tables 1–3 below, there is a considerable difference between the results which are carried out before the lectures and the results related to the second administration of the test (results related to the first administration of the test are always on the left for each comparison). On the one hand, it is possible to note that there are no significant differences (between the first administration of the test and the second one) related to Section 1 or Section 2. Moreover, in these sections, students have reached a score really closed to the maximum allowed value. On the other hand, it is not possible to say the same thing for Sections 3 and 4. In these cases, differences between the first administration of the test and the second one are very strong and they are in the direction to confirm the positive behavior of the proposed approach in terms of didactic efficiency.

The negative trend of the initial test results is mainly produced by the results of section 3 and section 4 which are based on questions that require a minimum level of knowledge at a conceptual level. The approach described above demonstrates, at the outcome of the test, its effectiveness since the same test, administered after only two lessons, produced completely different results. As it is possible to see, the difference is mainly due to the results of the questions posed in sections 3 and 4 which, in the preliminary test, provided the major difficulties to the students and had affected the final result.

Table 1. The mean score obtained by the students. The rows refer to the classes and the columns refer to the 4 different sections in which the tests have been divided. Columns for each section are double, "before" and "after" the cycle of lectures.

Class	Sec. 1 before /5	Sec. 1 after /5	Sec. 2 before /6	Sec. 2 after /6	Sec. 3 before /10	Sec. 3 after /10	Sec. 4 before /25	Sec. 4 after /25
A	3.14	3.52	3.86	4.56	1.1	9.71	2.71	24.33
B	1.64	4.45	3.86	5.77	1.82	9.45	0.27	22.5
C	3.08	4.83	4.8	4.63	0.32	9.75	0.68	23.96

Table 2. The standard deviations relative to the means listed in Table 1.

Class	Sec. 1 before /5	Sec. 1 after /5	Sec. 2 before /6	Sec. 2 after /6	Sec. 3 before /10	Sec. 3 after /10	Sec. 4 before /25	Sec. 4 after /25
A	2.41	1.01	2.89	1.28	1.23	0.39	3.06	0.7
B	6.23	0.79	8.66	0.36	2.6	0.43	0.74	1.25
C	1.19	0.22	2.24	1.98	6.37	0.35	1.26	1.96

6 Conclusions

The experimentation performed in Liceo Scientifico "Filolao" using the grossone-based methodology has shown that the symbol-concept approach can be an improved solution with respect to the traditional concept-symbol approach at least in the discussion of those topics that are intrinsically difficult to treat and understand. It has been shown that the ①-based methodology allows students to understand better concepts related to infinity and infinitesimals. The authors hope that this work can be a starting point for similar educational experiments carried out on a large scale and even in different initial conditions regarding periods of the test, age of the students of the sample, etc. In this way it is possible to have a better overview of the real effectiveness of this strategy in terms of educational impact. Starting from these considerations, the authors intend to identify new didactic paths for the introduction in teaching of new computational algorithms that make use of the acquired knowledge.

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Table 3. Table presents the mean vote of all the sections 1–4 with weights. The last two columns yield the relative standard deviations as in Table 2.

Class	Mean, sections 1-4 before lectures /46	Mean, sections 1-4 after lectures /46	Standard dev. before lectures /46	Standard dev. after lectures /46
A	10.81	42.52	9.58	4.82
B	7.59	42.1	29.7	1.48
C	8.88	43.17	11.63	4.81

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