Using Grossone to count the number of elements of infinite sets and the connection with bijections

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Abstract

In this paper, we look at how the notion of bijections can be used within the frame of Sergeyev's numeral system. We give two definitions for counting the number of elements of a set and we explore the connections between these two definitions. We also show the difference between this new numeral system and the results of the traditional naive set theory.

1 Introduction

This paper looks at a possible axiomatic foundation for the use of bijections in the new methodology introduced by Yaroslav SERGEYEV in his seminal papers, see [3, 4, 5] which we will refer to as the **new numeral system**. This system contains the standard numeral system to write finite integers, positive and negatives. It also contains a symbol, ①, which is, by definition the number of elements of the set of natural numbers with this property that $n <$ ① for any finite positive integer n. We refer the reader to [3, 5, 8] for more details and motivations on the system.

In Section 2, we look again at the notion of bijection in the traditional setting and, on an example, how it works in the new setting of the new numeral system.

In Section 3, we present a proposal toward a formalization within the frame of the new numeral system.

As we shall several times refer to the postulates of the new numeral system, we reproduce them here for the convenience of the reader, exactly as they are stated in [5, 8].

Postulate 1 *We postulate the existence of infinite and infinitesimal objects but accept that human beings and machines are able to execute only a finite number of operations.*

Postulate 2 *We shall not tell* **what are** *the mathematical objects we deal with; we shall just construct more powerful tools that will allow us to improve our capacities to observe and to describe properties of mathematical objects.*

Postulate 3 *We adopt the principle* 'The part is less than the whole' *to all numbers (finite, infinite and infinitesimal) and to all processes (finite and infinite).*

2 Bijections and the principle 'the part is less than the whole'

Remember that Cantor's set theory is based on the famous Bernstein theorem which states the following assertion:

Theorem 1 − (Bernstein) *Let A and B be sets such that there is an injective mapping f from A into B and an injective mapping g from B into A. Then, there is a bijection φ from A onto B.*

Traditionally, sets A and B such that there is a bijection from A onto B are called **equipotent**, we also usually say that they have the **same number** of elements. This relation between A and B is denoted by $A \equiv B$. If there is an **injection** from A into B, then it is said that A has no more elements than B and this is denoted $A \leq B$. Bernstein's theorem says that this latter relation defines an order among the sets.

Now, if we look at many examples of mathematical objects with the new tool given by the new numeral system, it seems that there is a blatant contradiction between the just mentioned theorem and Postulate 3 of the new numeral system, see [5]. Indeed, in traditional mathematics, to have a part as big as the whole is the characteristics of the infinite sets. With Postulate 3, this is no more true and a proper part of an infinite set is always less than the whole. However, in his seminal papers, Yaroslav Sergeyev always stresses that his new theory does not contradict Cantor's theory, that it simply gives new tools to better study infinite objects than those provided by traditional set theory.

We shall look at the following example taken from geometry using the traditional langage and the new one, see [5, 8, 10].

Let us start using the traditional point of view. Let A be a half-plane. Let δ be the line which is the border of A, see the left-hand side picture of Figure 1. Let C be the reflection of A in δ, next picture of Figure 1. Let h be another line of the plane, parallel to δ and inside A with $h \neq \delta$, third picture of Figure 1. Let B be the reflection of C into h, see the rightmost picture of Figure 1. Clearly, $B \subset A$ and $B \neq A$. According to Cantor's theory, A and B have the same number of elements. According to Sergeyev's system, as $B \subset A$ and $B \neq A$, B has less elements than A.
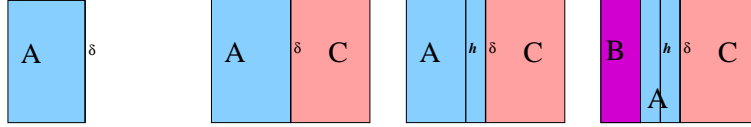
Figure 1 *An apparent contradiction between Cantor's theory and the new numeral system.*

Presently, let us apply the new approach as explained in [5, 8, 10]. Let us resume from the very presentation of the objects we consider. What is a half-plane? What is a half-line as, in this example, the really objects at work are half-lines. Traditionally, we would write a half-line as $]-\infty, a]$ or $[a, +\infty[$ where a is some real number. The new approach points at the fact that $-\infty$ and $+\infty$ are not precise notions. Compared with natural numbers, they are in the same position as *many* is with the natural numbers of Pirahã's[1], which are exactly 1 and 2, see [1, 3, 4, 5]. And so, Yaroslav continued, we have to precise the bounds of the infinite interval of which consists our half-line: $[a, b]$ where a and b are numbers, finite or infinite. And then, he continued, let us make the computations associated with the considered reflections.

Here, I provide this computation, in order the reader could appreciate what is found out. Let $A = [-b, a] \times I$, where b is a positive infinite number, a is the abscissa of the point where δ cuts the x-axis and $I = [-c, c]$ is an infinite interval with c a positive infinite number. As our reflections are performed in axes which are perpendicular to the x-axis, we perform the computations on abscissas only. The reflection in the line δ transforms x into $-x+2a$. And so, we get that $C = [a, b+2a] \times I$. Let d be the abscissa of the points where the line h cuts the x-axis. Similarly, the reflection in h transforms x into $-x+2d$, so that $B = [-b-2a+2d, -a+2d] \times I$. Now, it is plain that $-a+2d < a$ as we assume $d < a$ and that $-b-2a+2d < -b$, for the same reason. And this shows us that $B \not\subset A$, contrary to what was concluded from the example, see Figure 2. Note that the same computations performed in the frame of Cantor's theory shows that from $A =]-\infty, a] \times L$ with $L =]-\infty, +\infty[$, we get $C = [a, +\infty[\times L$ and $B =]-\infty, -a+2d] \times L$. Accordingly, as Cantor's theory does not allow us to distinguish between infinite quantities, we cannot see that the left-hand side bound of B is to the left-hand side of the left-hand side bound of A and so, there are infinitely many points of B which are not contained in A.

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[1]Papers [3, 4, 5], mention the discovery reported in [1] of an isolated group of people in Amazonia which have exactly the following numbers: 1, 2 and *many*.
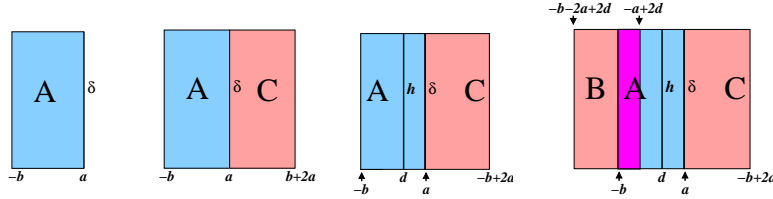
Figure 2 *No contradiction: with the new numeral system, the definition of the images is more precise.*

From this example, we can draw the conclusion: we have to make it as exact as we can the description of the objects on which we are working. In our example this concerns the sets we considered and the operations which we performed on these sets and, as far as numbers are involved, in which numeral system the numbers are expressed. More precisely, we had to indicate the exact bounds of our 'half-planes', it should be better to call them semi-infinite strips, *i.e. portions of the plane* in between two parallel lines. We also had to write down the transformations, here reflections in line, explicitly.

It is important to notice that the new numeral system is not a sub-system of non standard analysis and that it is neither a sub-system of the theory of ordinals. These points will be made more clear a bit further.

3 How to use bijections in the new numeral system

First, we look at the definition of the number of elements of a set and then, how to deal with this notion in connection with bijections.

3.1 The number of elements

The simplest way to define the number of elements of a set E is to count the number of its elements. Note that, practically, we can actually count finite sets only, moreover, with a rather small number of elements. This is contained in Postulate 1 of the new numeral system, see [5].

Now, if we wish to perform abstract considerations, we have to bypass this possibility and this is why we use descriptions. As long as we can describe an object, we may consider that we handle it, if needed. Of course, what we can do for descriptions strongly depends on the language we use to formulate them, this is the content of Postulate 2, see [5].

We have here to introduce a few notations which will be used in the statements and the proofs which we provide later. In particular, if μ is a number of the numeral system, we denote by $[1..\mu]$ the set of integers k which satisfy the relations $1 \leq k$ and $k \leq \mu$. Here and later, k or μ are any symbol used to define a number is assumed to design the representation of this number in the numeral system.

We start by defining how we can define the number of elements of a set.

Definition 1 *Let $\mathcal{S}$ be numeral system and let E be a nonempty set. We say that f is a **measurement** of E in $\mathcal{S}$ if there is a numeral in $\mathcal{S}$ expressing a positive integer μ, finite or infinite, with the property that f is a bijection of $[1..\mu]$ onto E. We say that a set E is **measured** in $\mathcal{S}$ if there is a measurement of f.*

When this is the case, we say that E is **measured** by $[1..\mu]$ and, for short, that it is measured by μ. We also say that μ is the number of elements of E. We shall also denote by $\sharp E$ the number of elements of E. Note that μ is required to be an integer only, that it may be finite or infinite and, in the latter case, that it is not restricted to be bounded by ①. Also note that when we say that E is measured, we have to be able to provide an f which measures E. Moreover, this f must be described, not merely assumed to exist. We use the words *measure*, *measured* and *measurement* because we have in mind the possibility to count the number of elements beyond ① if needed. As already mentioned about μ, it must be possible to write it in some numeral system. This means that a measurement can be performed only if we have at our disposal a numeral system allowing us to express the number μ used in Definition 1.

In Definition 1, we insist in the fact when a set is measured, the measurement refers to a numeral system $\mathcal{S}$ which has to be made explicit. As an example, the set $\{1, 2\}$ is measured in the Pirahãs' system as it is measured in the standard numeral system as well as in the new one and, in all these cases, by the identity function. However, the set $\{1, 2, 3\}$ cannot be measured in the Pirahãs' system where number 3 cannot be expressed. Of course, $\{1, 2, 3\}$ is measured in the standard numeral system as well as in the new one, again with the help of the identity function. This relativity is very important, it is contained in Postulate 2: if we have a more precise language we can see more properties.

Next, we consider how to define the number of elements of a subset of a set. Starting from this point, in order to make statements easier to read, we shall not repeat the reference to a numeral system. However, as indicated above, we have to consider that the word *measurement* always refers to such a system, as it must be possible to express the number used to establish the measurement in the numeral system for which the definitions or the theorems are applied.

Definition 2 *Let A be a subset of the set E. We say that A is **co-measured** in E if there are measurements f and g such that f measures A and g measures $E \backslash A$.*

Proposition 1 *If A is a co-measured subset of E, then E is measured. We have that $\sharp E = \sharp A + \sharp \{E \backslash A\}$.*

Proof. Let f be a measurement of A and let g be a measurement of $E \backslash A$. There are positive integers μ and λ such that f is a bijection from $[1..\mu]$ onto A and g is a bijection from $[1..\lambda]$ onto $E \backslash A$. Indeed, we define h from $[1..\mu+\lambda]$ as follows:

$$h(n) = \begin{cases} f(n) \text{ if } n \in [1..\mu] \\ g(n-\mu) \text{ if } n \in [\mu+1..\mu+\lambda] \end{cases}$$

■

We can also extend the definition of the number of elements of a set in the following conditions.

Theorem 2 *Let A be a measured set. Then B has the same number of elements as A if and only if there is a bijection from A onto B.*

Proof. Assume that there is a bijection f from A onto B. Let h be a measurement of A. Then $h \circ f$ is a measurement of B.

Conversely, let f be a measurement from $[1..\mu]$ onto A and g be an measurement of $[1..\nu]$ onto B. Then, as $\mu = \nu$, $g \circ f^{-1}$ is a bijection from A onto B.
■

Now, we can order the measured sets by their number of elements.

Theorem 3 *Let A and B be two measured sets. We have that $\sharp A \leq \sharp B$ if and only if there is an injection from A into B.*

Proof. Let f be a measurement from $[1..\mu]$ onto A and g be a measurement from $[1..\nu]$ onto B. Assume that $\mu \leq \nu$. Then $g \circ f^{-1}$ is an injection from A into B.

Conversely. Assume that there is an injection h from A into B. Assume that $\nu < \mu$. Then $h \circ f$ is a measurement from $[1..\nu]$ onto a proper subset C of B. Now, by the previous theorem, $\sharp C = \sharp B$. This is a contradiction with Postulate 3. Consequently, as the order on integers is linear, $\mu \leq \nu$. ■

Corollary 1 *Let A and B be two measured sets. Then if there is an injection from A into B and if there is an injection from B into A, then $\sharp A = \sharp B$.*

Proof. If the injection from A into B would not be surjective, we would obtain a proper subset of B with as many elements as B, a contradiction with Postulate 3.
■

And so, we can see that Bernstein's theorem is true for measured sets.

We have the following property which has no counter-part in the traditional theory:

Theorem 4 *Let A and B be two measured sets with $\sharp A = \sharp B$. Then, if $A \cap B \neq \emptyset$ and $A \neq B$, then $A \backslash (A \cap B)$ and $B \backslash (A \cap B)$ are nonempty sets. Moreover, if $A \cap B$ is co-measured in A and in B, then both $A \backslash (A \cap B)$ and $B \backslash (A \cap B)$ are measured, and $\sharp A \backslash (A \cap B) = \sharp B \backslash (A \cap B)$.*

Proof. Let A and B be sets satisfying the assumptions of the first sentence of the theorem. Let f be a measurement from $[1..\mu]$ onto A and g be a measurement from $[1..\mu]$ onto B, as $\sharp A = \sharp B$. If $A \cap B \neq \emptyset$ and $A \neq B$, as we cannot have $A \subset B$ thanks to $\sharp A = \sharp B$ and to Postulate 3, we have necessarily that $A \backslash (A \cap B) \neq \emptyset$. Similarly, from $A \neq B$ we cannot have $B \subset A$ and so,

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$B \backslash (A \cap B) \neq \emptyset$. Now, if $A \cap B$ is co-measured in both A and B we have three measurements f_A, h and g_B from $[1..\mu{-}\lambda]$ onto $A \backslash (A \cap B)$, from $[1..\lambda]$ onto $A \cap B$ and from $[1..\mu{-}\lambda]$ onto $B \backslash (A \cap B)$ respectively. The last assertion of the theorem follows from Theorem 3. ■

3.2 Defining numbers

Starting from this subsection and for the rest of the paper, we work in the new numeral system defined by Yaroslav Sergeyev, see [3, 4, 5]. It should be appropriate to extend the frame of Definition 1 to the definition of elements suggested in [5], Subsection 5.4. Let us illustrate this by the example of $\lfloor \sqrt{①} \rfloor$. This number is defined as the number of elements of the set $\{x \mid x^2 \leq ①\}$. In [5], Subsection 5.4, this is generalized from the function $x \mapsto x^2$ to any strictly increasing function g. When we consider a positive finite number n, we know that n is the number of elements of the set $[1..n]$. This definition is extended to any number, including ① and beyond. Now, consider an initial segment S of $[1..\kappa]$, where κ is some infinite positive integer. This means $1 \in S$ and that when $x \in S$ and $y < x$, we have $y \in S$. If $S \neq [1..\kappa]$, there is some number μ such that $\mu \notin S$. Note that if there is a number σ which is the number of elements of S, then $S \subset [1..\sigma]$, by definition of the measurement, here by the identity function. Now, if $\sigma \notin S$, then $S \subset [1..\sigma{-}1]$ which means that S has at most $\sigma{-}1$ elements, a contradiction. And so we have that $S = [1..\sigma]$. Consequently, if we consider that the number of elements of an initial segment of $[1..\kappa]$ is defined, then any initial segment of $[1..\kappa]$ is of the form $[1..\sigma]$ for some number $\sigma \leq \kappa$. But this is an assumption so that we have to formulate it as an axiom:

Axiom 1 *Let κ be a positive integer finite or infinite, possibly greater than ①. For each initial segment S of $[1..\kappa]$, there is a number $\sigma \leq \kappa$ such that $S = [1..\sigma]$. Clearly, σ is the **number of elements** of S.*

There is here a difference between the classical theory of sets. Consider the set S_n of positive integers k which are less than n. Clearly, S_n is an initial segment when $n > 1$. In the traditional theory, $\overset{\infty}{\underset{n=1}{\cup}} S_n = I\!N$. Using the new numeral system, we get that $\underset{n \leq ①}{\cup} S_n = [1..①{-}1]$.

Now, we can see that the way we defined $\lfloor \sqrt{①} \rfloor$ is legitimated by Axiom 1. We may also define $\lfloor \sqrt{\kappa} \rfloor$ and many other numbers can be defined in this way as $\lfloor \log ① \rfloor$ and $\lfloor \log_b ① \rfloor$ for any positive finite number $b \geq 2$. However, note that, according to Postulate 1, the number of applications of Axiom 1 is finite so that in fact, we enlarge the numeral system by introducing such notations, only finitely many times and in a finite way.

It is important to repeat that Axiom 1 applies to sets which are measured and so it must be possible to express the number σ in the same numeral system $\mathcal{S}$ as the one used to express the number κ. This means that Axiom 1 does not apply to any set. In particular, we cannot apply the axiom to the set of finite positive integers. The main reason is that this set is not well described, even in the new

numeral system introduced by Yaroslav Sergeyev. According to Postulate 2, we cannot say what are the finite positive integers, so that we cannot speak about their set with precision. However, we can speak of the set $\mathcal{F}_\mathcal{S}$ of the expressions in $\mathcal{S}$ of the finite positive integers. This set is clearly an initial segment of $[1..①]$ so that there is a finite positive integer $\varphi_\mathcal{S}$ expressible in $\mathcal{S}$ such that $\mathcal{F}_\mathcal{S} = [1..\varphi_\mathcal{S}]$ and such that $\varphi_\mathcal{S}+1$ is not expressible in $\mathcal{S}$.

Note that this shows us that the Peano axiom works on another plan: it says that if n is a positive integer, so is $n+1$. But in stating such a property, this axiom does not consider as relevent the possibility to express both n and $n+1$. To say things in other words, Peano axiom does not take into account practical limitations in expressing numbers concretely. Now, Postulates 1 and 2 tell us that we always use a language to describe objects and that the quality of the description depends on the expressive power of the language. In particular, we have to take into account the limitations on writing the expression of a positive integer.

In any numeral system $\mathcal{S}$, there is a maximal integer $\varphi_\mathcal{S}$ which can be expressed in $\mathcal{S}$. This limitation is not that surprising: in most practical programming languages, there is a constant `maxint` which denotes the greatest positive integer. The operation `maxint`+1 cannot be performed and if your program performs such an operation, polite compilers inform you that there was an attempt to use a non admissible value for the indicated type. Accordingly, the application of Axiom 1 to the expressions of finite positive numbers is very natural. This stresses the usefulness of Axiom 1.

Note that Axiom 1 is in full agreement with Postulate 3 saying that *the part is less than the whole* is always true, whatever the sets while this principle does not hold for Cantor's infinite cardinals. Also note that in the traditional ordinal theory, ω and $\omega+1$ exist but $\omega-1$ cannot be defined as ω is a limit-ordinal.

As a corollary of Axiom 1 we can state the following property:

Theorem 5 *Let κ be a positive integer, finite or infinite, and let g be a strictly increasing function over $[1..\kappa]$. Let μ be the number of elements of the set $G = \{x \mid g(x) \leq \kappa\}$. We know that $G = [1..\mu]$. The number μ is also characterized as the single number x such that $g(x) \leq \kappa < g(x+1)$.*

Proof. As $\mu \in G$, $g(\mu) \leq \kappa$. If we do not have $\kappa < g(\mu+1)$, then we have $g(\mu+1) \leq \kappa$ so that $\mu+1 \in G$ and, as g is strictly increasing, G is a segment and it contains $[1..\mu+1]$, a contradiction with the definition of μ. And so, $\kappa < \mu+1$. The uniqueness of μ follows from the fact that g is strictly increasing. ∎

Note that $g(\mu+1) - g(\mu)$ may be infinite: when $g(x) = x^2$, we have that $g(\mu+1) - g(\mu) = 2\mu+1$ and if μ is defined by the characterization of Theorem 5 with $\kappa = ①$, μ is clearly infinite.

Let us remark that Axiom 1 and Theorem 5 allow us to define a lot of numbers: this is a general paradigm related to the notion of definability. However, concretely, we may apply them to finitely many instances of concrete formulas only so that these new tools remain in agreement with Postulate 1 of Sergeyev's new system.

Now, Axiom 1 allows us to define an important notion: that of the smallest element of a set. Namely,

Theorem 6 *Let κ be a positive integer, finite or infinite and let A be a non-empty set of $[1..\kappa]$. Then, there is an integer μ in $[1..\kappa]$ such that $\mu \in A$ and for any $n \in A$, $\mu \leq n$.*

Proof. Let S be the set of x in $[1..\kappa]$ such that for any $n \in A$ $x \leq n$. If $1 \in A$, we have the smallest element of A. And so, assume that $1 \notin A$. Clearly, S is nonempty and S is a segment of $[1..\kappa]$. And so, from Axiom 1, there is an integer $\nu \in [1..\kappa]$ such that $S = [1..\nu]$. If $\nu \in A$, we are done. Now if $\nu \notin A$, then for all $n \in A$, $\nu+1 \leq n$. But then, $\nu+1 \in S$, a contradiction with the definition of ν. ■

The smallest element of A is denoted by $\min A$.

Theorem 6 allows us to prove a stronger version of Theorem 4. We start with the following property.

Theorem 7 *Let E be a measured set and let A be a measured subset of E. Then A is co-measured in E.*

Proof. We may assume that A is a proper nonempty subset of E. Let f be a measurement of $[1..\mu]$ onto E and let g be a measurement of $[1..\lambda]$ onto A. Then, we define $h(1) = \min(E \backslash A)$ and, from this, $A_1 = E \backslash (A \cup \{h(1)\})$. We define $h(n+1) = \min(E \backslash A_n)$ and from that, similarly, $A_{n+1} = E \backslash (A_n \cup \{h(n+1)\})$. Let S be the set of n such that $h(n)$ and A_n are defined. It is clearly an initial segment of $[1..\nu]$ for some $\nu \geq \mu-\lambda$. And so, it has a greatest element π. It is plain that $A_\pi = \emptyset$. Otherwise, we could define $h(\pi+1)$ and $A_{\pi+1}$, a contradiction. Now, as $A_\pi = \emptyset$, this proves that $\pi \geq \mu-\lambda$ as h is injective. Now, as $h([1..\pi]) \subseteq (E \backslash A)$ by construction, $\pi \leq \mu-\lambda$, so that $\pi = \mu-\lambda$ and h is surjective. ■

Corollary 2 *Let A and B be two measured sets with $\sharp A = \sharp B$. Then, if $A \cap B \neq \emptyset$, $A \cap B$ is measured and $A \neq B$, then $A \backslash (A \cap B)$ and $B \backslash (A \cap B)$ are measured nonempty sets. Moreover, $\sharp A \backslash (A \cap B) = \sharp B \backslash (A \cap B)$.*

Proof. From Theorem 7, $A \cap B$ is co-measured in both A and B. So that Theorem 4 applies. ■

We have another important result:

Theorem 8 *Let κ be a positive integer, finite or infinite. Let A be a non empty set of $[1..\kappa]$. Then A is measured.*

Proof. We repeat the argument of Theorem 7. Let A be a non-empty set of $[1..\kappa]$. Then, we know from Theorem 6 that A has a smallest element. Define $f(1) = \min A$ and define $A_1 = A \backslash \{f(1)\}$. Define for any positive n, finite or infinite: $f(n+1) = \min A_n$ and $A_{n+1} = A_n \backslash \{f(n+1)\}$. Let S be the set of

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$x \in [1..\kappa]$ such that f is defined on $[1..x]$. As S is non empty, the above application of Theorem 6 shows us that $1 \in S$. Now, it is pain that if $x \in S$ and $y \in [1..\kappa]$ with $y \leq x$, then $y \in S$. So that S is a non empty segment of $[1..\kappa]$. From Axiom 1, there is an integer $\mu \in [1..\kappa]$ such that $S = [1..\mu]$. Now, $A_\mu = \emptyset$, otherwise, $f(\mu+1)$ could be defined, and then $\mu+1 \in S$, a contradiction with the definition of μ. Now, f is injective by construction and, by the construction of f, as $A_\mu = \emptyset$, f is surjective onto A. And so, f is a measurement of A. ■

It is important here to remind the reader that Theorem 8 deals with sets which are clearly described only. This is why the theorem says "Let A be a non empty set of..." and not "for any non empty set of...". We have to also remark that in most cases of a concrete set, the measurement is given with the description of the set. Also, we remind the reader that the number used to measure a set has to be expressed in an explicit numeral system. According to Postulate 1, we can only perform finitely many operations on finitely many objects. Accordingly, each time we apply Theorem 7, its corollary and Theorem 8, we can give appropriate expressions: we use only finitely many symbols.

As an example of a set for which we cannot immediately number its elements, we can indicate the set of **prime** integers, were a positive number greater than 1, finite or infinite is prime whether it has two divisors exactly: 1 and itself. This set is clearly infinite but, at the present moment, we cannot say that it is measured and, also, we cannot prove that it cannot be measured.

Now, let us consider an infinite positive integer κ, and let us consider the transformation $\iota : x \mapsto \kappa+1-x$. It maps $[1..\kappa]$ onto itself and it is clearly a bijection as it is involutive. Now, it is easy to see that if $x, y \in [1..\kappa]$, then $x < y$ if and only if $\iota(x) > \iota(y)$. This allows us to state the following property:

Theorem 9 *Let κ be an infinite positive integer and let A be a non empty subset of $[1..\kappa]$. Then A contains an element x such that for any $y \in A$, $y \leq x$. We say that x is the greatest element of A and it is denoted by $\max A$.*

Proof. Let $\overline{A}$ be the image of A under ι. Then, as ι maps $[1..\kappa]$ onto itself, by Theorem 6, $\overline{A}$ has a smallest element m. Let $x = \iota(m)$. For $y \in A$, we get $\iota(y) \geq m$ and so, $x = \iota(m) \geq \iota(\iota(y)) = y$. Accordingly, x is the greatest element of A. ■

Now, we can define the notion of final segment:

Definition 3 *Let κ be an infinite positive integer. Say that a nonempty subset F of $[1..\kappa]$ is a **final segment** if $\kappa \in F$ and, for any x in F and any $y \in [1..\kappa]$, from $x \leq y$, it follows that $y \in F$.*

Now, it is clear from this definition that A is a final segment of $[1..\kappa]$ if and only if $\overline{A} = \iota(A)$ is an initial segment of $[1..\kappa]$. We obtain:

Theorem 10 *Let κ be an infinite positive integer. A nonempty subset F of $[1..\kappa]$ is a final segment of $[1..\kappa]$ if and only there is an integer $\nu \in [1..\kappa]$ such that $F = [\nu..\kappa]$.*

Proof. Apply Axiom 1 to $\overline{F}$ and then, apply again ι as $\iota(\overline{(F)}) = F$. ■

Accordingly, any nonempty subset A of $[1..\kappa]$ has a smallest element and a greatest one. This allows us to define the **convex hull** of A for any non-empty set A of $[1..\kappa]$.

Definition 4 *Let κ be an infinite positive integer. A nonempty subset A of $[1..\kappa]$ is **convex** if and only if for any $x, y \in A$ with $x \leq y$, then $[x..y] \subset A$. If A is any non-empty subset of $[1..\kappa]$, its **convex hull** is the smallest convex subset included in $[1..\kappa]$ which contains A.*

Theorem 11 *Let κ be an infinite positive integer. Let A be a non empty subset of $[1..\kappa]$. Then A has a convex hull which is $[\min A..\max A]$.*

3.3   Discussion

We would like to discuss a few points about the results of this paper.

First of all, remember that in Subsection 3.2, we have considered κ as an infinite positive integer. We have mentioned after Definition 1 that we may consider infinite integer which are greater than ①. In Yaroslav Sergeyev's works, it was several times indicated that there are sets whose number of elements are greater than ①. Let us give the following examples given in [6, 7, 8]. The set of integers, $\mathbb{Z}$, has 2①+1 elements. The set $P = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{N}\}$ has ①^2 elements and the set of numeral expressions of the form $(.a_1 a_2 ... a_{①})_b$ with $0 \leq a_i < b$ has $b^{①}$ elements. Now, as pointed at in [7], the word *sequence* is restricted to subsets of $[1..①]$ as well as the words *enumerate* and *enumeration*. This is why in Section 3 we used the words *measure*, *measured* and *measurement*. Now, we did not use the term *measurable* which is used in mathematics in a completely different environment. The words *measure*, *measured* and *measurement* refer to one of the historically first physical process. The idea is to stress on the concreteness of the notion: it evokes comparison with a yardstick to measure the length of objects. Here we have the same idea of comparison with a yardstick: the set of numbers up to a given one. Now, there is another reason why the word *measured* is used instead of *measurable*. When we say that a word is *measured* we have always to have in mind how it has been measured, *i.e.* we have to know at least one way to do that, and so, we have to know at least one *measurement* of the initial segment of $[1..\kappa]$ which can be put in bijection with the set, as well as to be able to write κ in the numerical system we use and the measurement explicitly describes the bijection.

Second, it is again a point which we have already stressed: when we speak of a set and of an application on the set, we know a description of the set and a description of the application. As already mentioned, the description depends on our language. We have already indicated how the introduction of ① allows us to distinguish much more clearly between infinite sets than with the traditional Cantor theory which cannot see any difference in the number of elements between for instance $\mathbb{N}$ and the set of pairs of positive natural numbers.

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Now, this remark is very important. We have mentioned that Pirahãs have only three numbers 1, 2 and *many* and that the computation rules involving *many* and 1 or 2 are very similar to Cantor's rules involving ∞ and finite natural number. What we have to stress here is that this difference is very important. The new tools allow to see better, but the previous tools cannot see what is seen by the new ones. As an example, see [8], Pirahãs cannot define the set $\{1, 2, 3, 4, 5\}$. They can define the first two elements but the three others have no meaning for them. And so, many problems about infinite sets which are formulated in the frame of Cantor's theory have a new formulation in the new numeral system and for some of them, the problem simply vanishes. In particular, we refer the reader to [6, 9] for important results in this regard.

Conclusion

It seems to me that this paper stresses in a right way the importance of being precise when looking at the number of elements of sets, especially when we wish to compare them in this regard. We have to look at well defined sets and, when comparing them, we also have to look at the tools on which the comparison relies.

It seems to me that with the material given in this paper, we have more tools to compute the number of elements of a set in the new numeral system devised by Yaroslav Sergeyev. I hope that this might contribute to new developments of this beautiful system.

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