Fibonacci words, hyperbolic tilings and grossone

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Abstract

In this paper, we study the contribution of the theory of grossone to the study of infinite Fibonacci words, combining this tool with the help of a particular tiling of the hyperbolic plane : the tiling $\{7, 3\}$, called the heptagrid. With the help of the numeral system based on grossone, we obtain a richer family of infinite Fibonacci words compared with the traditional approach.

ACM-class: F.2.2., F.4.1, I.3.5
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In [12], it was shown that a few languages constructed from some figures of hyperbolic tilings cannot be recognized by pushdown automata but they can be recognized by a 2-iterated pushdown automaton. Before, it was known that several tessellations of the hyperbolic plane are generated by substitutions, see [3]. This property is also clear from [9]. Some considerations of these properties from the point of view of fomal language grammars were also devised in [14]. In that paper, a new look on infinite Fibonacci words was introduced, using particular tessellations of the hyperbolic plane. The application of grossone to the study of hyperbolic tessellations of the hyperbolic plane was initiated in [13]. In this paper, we suggest to continue looking at this topic, this time at what can be said on infinite Fibonacci words.

We refer the reader to [13] for a short introduction to what has to be known about hyperbolic geometry. We also refer the reader to Yaroslav Sergeyev's foundational papers for what has to be known about grossone, see for instance [16, 17, 18] and also [7] for a general overview and further references.

In Section 1, we sketchily remind the very basic knowledge required about hyperbolic geometry and grossone. In Section 2, we remind what is known about infinite Fibonacci words in hyperbolic tessellations. In Section 3, we apply the theory of grossone in order to obtain a new look at these results.

1 Introduction

This section is devided into two sub-sections. The first one, Sub-section 1.1, reminds the reader about several features and properties on tilings of the hyperbolic plane. The second one, Sub-section 1.2 reminds what we have basically to know about grossone.

1.1 The tilings of the hyperbolic plane

We assume that the reader is a bit familiar with hyperbolic geometry, at least with its most popular models, the Poincarés's half-plane and disc. The reader is referred to [10, 11] for more details and useful references. We just remind him/her that in the disc model which we consider here, the hyperbolic plane is represented by a fixed open disc $\mathcal{D}$ of the Euclidean plane, as illustrated by Figure 1.

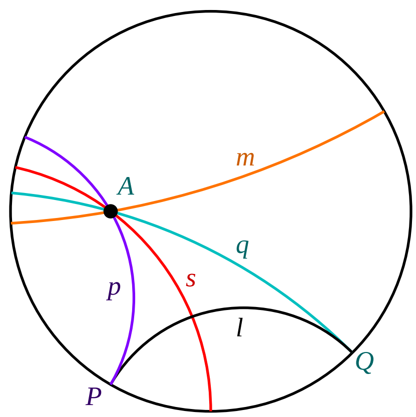

Figure 1 *Poincaré's disc model.*

In the model, lines are the trace in $\mathcal{D}$ either of a diameter or a cirle which is orthogonal to $\partial\mathcal{D}$, the border of $\mathcal{D}$. In the figure, we can see a point A out of a line ℓ whose Euclidean support cuts $\partial\mathcal{D}$ at P and D which, as points of $\partial\mathcal{D}$ are called **points at infinity**. It is important to remind that these points do not belong to the hyperbolic plane although they play an important role with many properties of the plane. As shown in the figure, some lines passing through A, as s, cut l in the hyperbolic plane. Some others, like m, do not cut it at all, neither inside $\mathcal{D}$ nor elsewhere, they are called **non-secant** with l. At last and not the least, two lines, p and q have exactly one point at infinity in common with l, P and Q respectively: they are called **parallel** to l.

Next, we remind the reader that in the hyperbolic plane, thanks to a well known theorem of Poincaré, there are infinitely many tilings which are generated by

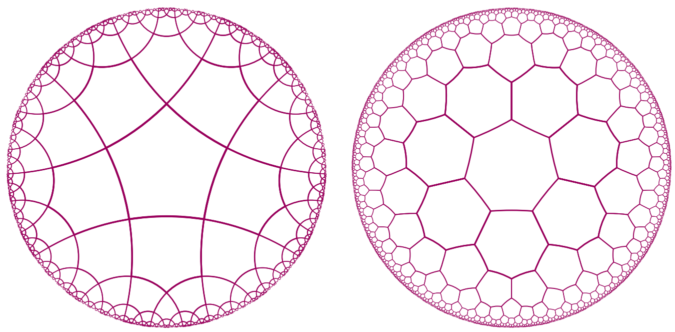

Figure 2 *Left-hand side: the pentagrid. Right-hand side: the heptagrid.*

tessellation starting from a regular polygon. This means that, starting from the polygon, we recursively copy it by reflections in its sides and of the images in their sides. This family of tilings is defined by two parameters: p, the number of sides of the polygon and q, the number of polygons which can be put around a vertex without overlapping and covering any small enough neighbourhood of the vertex.

In order to represent the tilings which we shall consider and the regions whose contour word will be under study, we shall make use of the Poincaré's disc model. Our illustrations will take place in the **pentagrid** and the **heptagrid**, *i.e.* the tilings $\{5,4\}$ and $\{7,3\}$ respectively of the hyperbolic plane. Figures 2 and 3 illustrate these tilings.

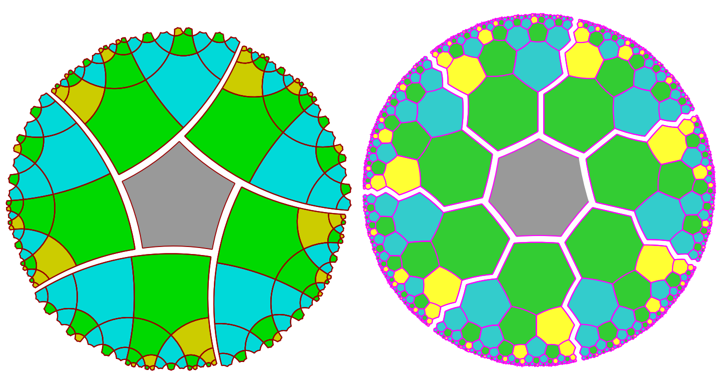

Figure 3 *Left-hand side: the pentagrid. Right-hand side: the heptagrid. Note that in both cases, the sectors are spanned by the same tree.*

From Figure 2, the pentagrid and the heptagrid seem rather different. However, there is a tight connection between these tilings which can be seen from Figure 3. In both pictures of the latter figure, we represent the tiling by selecting a central tile and then, by displaying as many sectors as the number of sides of the central tile. In each case, these sectors do not overlap and their union together with the central cell gives the tiling of the whole hyperbolic plane. Now, there is a deeper common point: in both cases, each sector is spanned by a tree which we call a Fibonacci tree for a reason which will soon be explained.

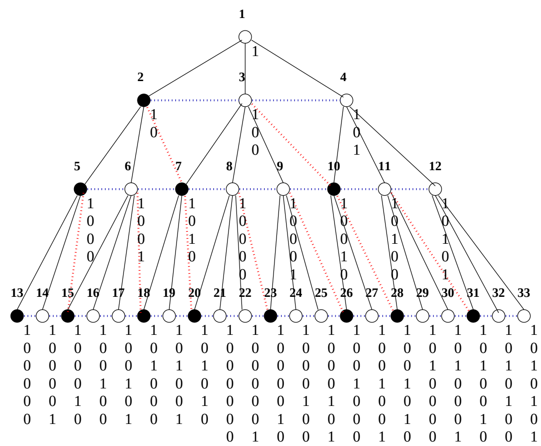

Figure 4 *The standard Fibonacci tree. The nodes are numbered from the root, from left to right on each level and level after level. For each node, the figure displays the representation of the number of the node with respect to the Fibonacci sequence, the representation avoiding consecutive* 1*'s. The doted lines complete the tree into the dual graph of the tiling, as they complete the numer of arcs at a node to seven. The vertical binary numbers are the representations of the numbers given to the nodes in the Fibonacci basis.*

As proved in [8, 10], the corresponding tree can be defined as follows. We distinguish two kinds of nodes, say black nodes, labelled by B, and white nodes, labelled by W. Now, we get the sons of a node by the following rules: $B \rightarrow BW$ and $W \rightarrow BWW$, the root of the tree being a white node, see Figure 4. It is not difficult to see that if the root is on level 0 of the tree, the number of nodes on the level k of the tree is f_{2k+1}, where $\{f_k\}_{k \in I\!N}$ is the Fibonacci sequence with $f_0 = f_1 = 1$.

The Fibonacci tree has a lot of nice properties which we cannot discuss here. In particular, there is a way to locate the tiles of the pentagrid or the heptagrid very easily thanks to coordinates devised from the properties of the Fibonacci tree, see [8, 10, 11].

1.2 The new numeral system

In papers [16, 17, 18], Yaroslav Sergeyev gives the main arguments in favour of the new numeral system he founded, allowing to obtain more precise results on infinite sets that what was obtained previously.

We can sum up the properties of the system as follows.

We distinguish the **objects** of our study from the **tools** we use to **observe** them. These three parts of the knowledge process have to be more clearly distinguished as they were traditionally in mathematics, contrarily to other domains of science, as physics and natural sciences where this distinction is clearly observed. This is the content of *Postulate* 2 in the quoted papers. It is an important issue for mathematics where the distinction between an **observer** and what is **observed** is very often forgotten. In particular, not enough attention is paid to subjectivity of the observer and the relative validity of his/her observations. The latter are very dependent of cultural elements, especially the *language* used by the observer to describe what he/she sees.

We are interested in the properties of the objects, some of them being possibly infinite or infinitesimal, but **operations** on the objects, performed by a human being or a machine, necessarily deal with **finitely many** of them and only **finitely many** operations can be applied within the frame of an argument. This is the content of *Postulate* 1 in the quoted papers.

At last and not the least, we consider that the principle *The part is less than the whole* has to be applied to all numbers, finite, infinite or infinitesimal, and also to all sets and processes, whether finite or infinite. This is the content of *Postulate* 3 of [16, 17, 18].

On the basis of these principles, Yaroslav Sergeyev introduced a new numeral system in order to be able to write down infinite numbers. To this aim, an **infinite natural number** is introduced, **grossone**, denote by $①$, which is the number of elements of the set of positive integers. This number satisfies the following three properties which are axioms of the system:

− for any finite natural number n, $n < ①$.

− we have $0 \cdot ① = ① \cdot 0 = 0$, $① - ① = 0$, $\frac{①}{①} = 1$, $①^0 = 1$, $1^{①} = 1$ and $0^{①} = 0$.

− let $I\!N_{k,n}$ be the set of positive integers of the form $k+jn$ for j running over the set of the positive integers; notice that these sets are pairwise disjoint and that their union is the set of all positive integers; then all these sets have the same number of elements denoted by $\frac{①}{n}$.

Denote by $I\!N$ the set of positive natural numbers. All traditional operations performed on natural numbers are extended to $①$ in a natural way with the standard properties, among them: commutativity and associativity of addition and mutiplication and distributivity of multiplication over addition. As n is the number of elements of the finite positive integers not greater than n, and as $①$ is, by definition, the number of elements of $I\!N$, a consequence of the properties of addition and multiplication, is that $I\!N$ also contains a lot of other infinite numbers: all of them of the form $\frac{①}{n}$ and, more generally, all the numbers

$\dfrac{j①}{n}\pm k$ for any $j \in 1..n$ and any finite natural number k, n being any positive integer.

From now on, we shall call the system described above: **infinite numeral system**. We conclude this short introduction to the infinite numeral system by a brief mention on **sequences** as we shall use this notion. A **sequence** of elements of a set A is a mapping from the set of positive integers into a set A.

Presently, we have all tools in order to look at the goal of the paper.

Before turning to this point, our application of this system to hyperbolic geometry and a particular aspect connected with the Fibonacci sequence and infinite Fibonacci words, we think it useful to give the reader many references which indicate the large spectrum of fields to which this new numeral system gives new insights, for example in cellular automata, in percolation phenomena, Turing machines, biology and differential equations, see [1, 5, 19, 21] and [20] respectively.

2 Infinite Fibonacci words in the heptagrid

From what we have indicated in Sub-section 1.1, the reader may guess that we shall use the above tilings of the hyperbolic plane in order to look at infinite Fibonacci words. Classically, such words are generated by the following substitution rules: $b \rightarrow a$ and $a \rightarrow ab$. In fact, our adaptation to the hyperbolic plane leads us to consider another couple of substitutions: $b \rightarrow ba$ and $a \rightarrow baa$. In Sub-section 2.1, we define a natural setting to give rise to Fibonacci words. Then, in Sub-section 2.2, we look at several ways to generate infinite Fibonacci words.

2.1 Contour words and words along a level

In [12], the author considered the possibility to define words by looking at a specific object: the set of tiles which lie at a given distance from another tile, fixed in advance and once for all.

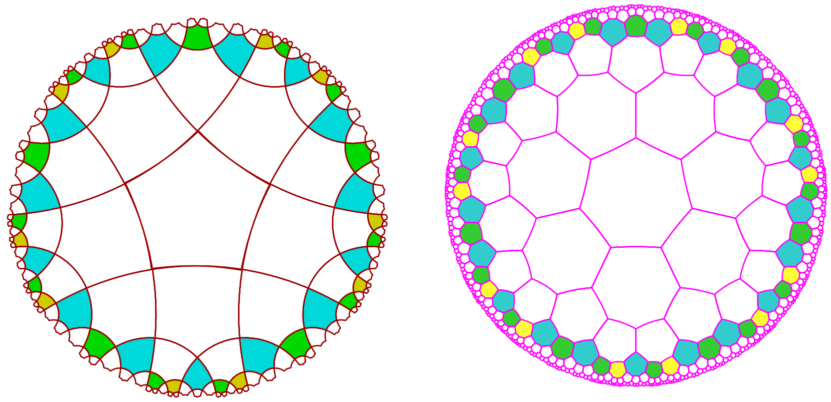

Figure 5 *Levels: to left, in the pentagrid; to right, in the heptagrid.*

Fix a tile C which will later be called the central one. A path from a tile T to C is a finite sequence T_i, $0 \leq i \leq n$ of tiles such that $T_0 = C$, $T_n = T$ and for all i in $[0..n-1]$, $T_i \cap T_{i+1}$ consists of one edge exactly. Then we say that n is the lenght of the path. The distance from T to C is the shortest length for a path joining T to C. Clearly, the distance is always defined. Now, a ball B around C of radius ρ is the set of tiles T whose distance to C is at most ρ. The border of B, centered at C and denoted by ∂B is the set of tiles whose distance to C is the radius of B.

Let D_n be the set of tiles which are at distance v from C. It was proved in [10] that the set of tiles which are on the same level in a Fibonacci tree belong to a part of the border of a ball around the root of the Fibonacci tree. It was also proved there that the number of tiles in D_n is $5f_{2n+1}$, $7f_{2n+1}$ in the case of the pentagrid, heptagrid respectively. The reader may understand the reason of this result from our study of Sub-section 1.1.

Let us imagine that we put letter B on the tiles of D_n which correspond to black nodes in the Fibonacci tree spanning their sector, and that we put the letter W on those which correspond to white nodes. Starting from the left-hand side branch of a fixed Fibonacci tree, we can define the words φ_n from $\{B, W\}^*$, the set of all words containing B and W only, which consist on getting the letters on the tiles of D_n one by one and putting them into a word in the same order as they are found on D_n. The interesting property is that the set of φ_n's cannot be recognized neither by a finite automaton, nor by a finite pushdown automaton, see [12]. As proved in that latter paper, the set of φ_n's can be recognized by an iterated pushdown automaton, see due to [2] for this notion and references. The reader may look at [12] for a proof of these results about the set of φ_n's.

2.2 Constructing infinite Fibonacci words

Without restricting the generality of the results, starting from this sub-section, we illustrate the results by figures in the heptagrid. The construction which will be now described was already performed by the author in [14] for what is illustrated by Figures 6 and 8. Here too, the construction is explained with more details. As in this sub-section, the arguments developped there take place within the classical approach to infinity. In that paper, we considered the substitution rules and the mechanism of their application within the frame of a **formal grammar**. For the commodity of the reader, we reproduce here the two grammars which are defined in [14]:

(G_1)	(G_2)
symbols: X, Y, Z, C,	symbols: X, Y, Z, C, W, B,
with C, W and B being terminals;	with C, W and B being terminals;
initial symbol: Z;	initial symbol: Z;
rules:	rules:
$Z \Rightarrow CY^\alpha$	$Z \Rightarrow CY^\alpha$
$Y \Rightarrow WXYY$	$Y \Rightarrow WYXY$
$X \Rightarrow BXY$	$X \Rightarrow BXY$

where $\alpha = 5$ for the pentagrid and $\alpha = 7$ for the heptagrid. As proved in [14], these grammars generate the pentagrid and the heptagrid. Consider again a fixed ball around C and fix one of the finite Fibonacci trees generated around C, say $\mathcal{F}$. We can imagine C as the central tile in Figure 6. Let B the ball around C which contains $\mathcal{F}$ and whose border contains the leaves of $\mathcal{F}$. It is not difficult to find a tile C_1 which is a neighbour of C and such that C is the root of a Fibonacci tree $\mathcal{F}_1$ in the ball B_1 around C_1 containing $\mathcal{F}$. We can assume that, in the same way, the border of B_1 contains the leaves of $\mathcal{F}_1$.

In Figure 6, left-hand side picture, we have a line δ_1 which passes through the mid-points of consecutive edges of heptagons. We define C_1 as the blue neighbour of C which is cut by δ_1 and which is above δ_1. We can remark that C is the image of C_1 by a shift along the line δ_1. Now, it is not difficult to see that the restriction of the tiling to $\mathcal{F}_1$ contains the restriction of the tiling to $\mathcal{F}$. We can also see that the leaves of $\mathcal{F}$ are contained in those of $\mathcal{F}_1$. In the left-and side picture of Figure 6, the sector generated by a black tile is delimited by the ray a and the line δ_1. The two sectors generated by a white tile are delimited by the line δ_1 and the ray e and then by the ray e and the ray b. A similar convention is followed for the tree rooted at C_1: the rays a_1, b_1 and e_1 play the same role for C_1 as the rays a, b and e respectively for C.

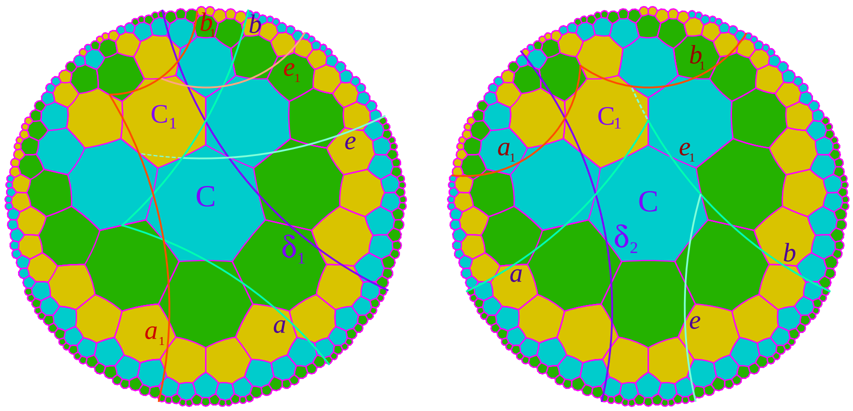

Figure 6 *Heptagrid: construction of bi-infinite words.*
Left-hand side: the bi-infinite word associated with the grammar (G_1)*. Right-hand side: the bi-infinite word associated with the grammar* (G_2)

From the figure, it is not difficult to see that, by induction, we construct a sequence of tiles C_n with C_0 a tile crossed by δ_1 and which is fixed once and for all, C_{n+1}, $n \geq 0$, is the neighbour of C_n which is crossed by δ_1 and which is defined by the fact that its distance from C_0 is $n+1$. We define B_n as the ball around C_n whose border contains C_0 and F_n is the Fibonacci tree rooted at C_n whose leaves are on B_n. This allows us to define a sequence of words w_n which is the trace of the leaves of F_n: w_n is in $\{B, W\}^\star$ and the j^{th} letter of w_n is B, W, depending on whether the j^{th} leave of F_n is black, white respectively. The construction shows us that w_n is a factor of w_{n+1} and we may assume that there are nonempty words u_n and v_n wuch that $w_{n+1} = u_n w_n v_n$.

A closer look at the substitutions we mentioned in Section 2 shows that $v_n = w_n$ and that $u_{n+1} = u_n w_n$. Indeed, the separation between u_n and w_n is materialized by δ_1. Note that the separation between the two occurrences of w_n is not fixed: it moves and tends to infinity as the length of w_n itself tends to infinity. And so, w_n is defined at the same time as u_n by the two equations:

$$\begin{aligned} w_{n+1} &= u_n w_n w_n \\ u_{n+1} &= u_n w_n \end{aligned} \qquad (E_1)$$

with initial conditions $u_0 = B$ and $w_0 = W$.

As the lengths of u_n and w_n tend to infinity, and as δ_1 is fixed, we can see from the left-hand side picture of Figure 6 that the sequence of words w_n tend to a bi-infinite word, *i.e.* a word whose both ends tend to infinity.

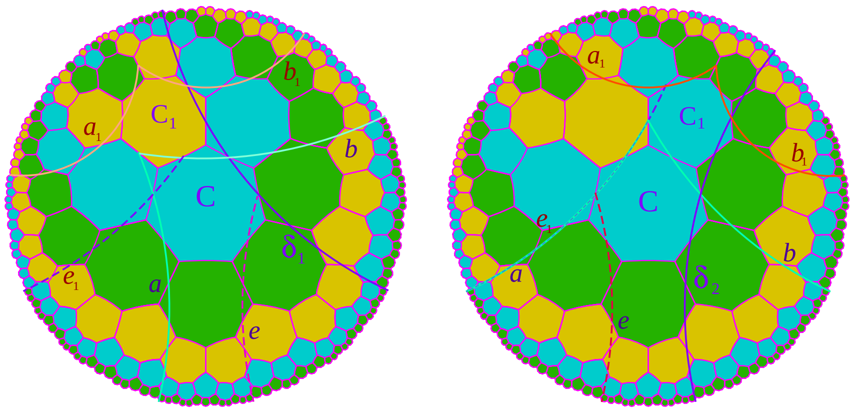

Figure 7 *Heptagrid: construction of another way to construct bi-infinite words.*
Left-hand side: the bi-infinite word associated with the grammar (G_1)*. Right-hand side: the bi-infinite word associated with the grammar* (G_2) *In the latter picture, note that* e_1 *and* a *are supported by the same Euclidean line.*

Define $\bigotimes\limits_{i=a}^{n+1} m_i = m_{n+1}\bigotimes\limits_{i=a}^{n} m_i$ and $\bigotimes\limits_{i=a}^{a-1} m_i = \epsilon$, where ϵ is the empty word. From (E_1), we easily derive that

$$w_{n+1} = \Big(\bigotimes_{i=1}^{n+1} u_i\Big) w_0. \qquad (C_f)$$

In the right-hand side picture of Figure 6, we have a similar picture by the trace of the leaves of the tree constructed according to the rules of (G_2) with $x_0 = B$ and $y_0 = W$. In the figure, we follow the same notations for the right-hand side picture as for its left-hand side part. Then, the equations satisfied by x_n and y_n are:

$$\begin{aligned} y_{n+1} &= y_n x_n y_n \\ x_{n+1} &= x_n y_n \end{aligned} \qquad (E_2)$$

Note that these words are very different from the w_n's and the u_n's. We remark that w_n and y_n have the same length and that, similarly, u_n and x_n also have the same length. Now, the first letter of w_n is B while it is W for y_n. The last letter of u_n is W while the last letter of x_n is B.

As we have argued with (E_1) for establishing (C_f), it is not difficult to see that from (E_2), we easily derive that

$$x_{n+1} = x_0\Big(\bigotimes_{i=1}^{n+1} y_i\Big). \qquad (C_g)$$

We can see that, this time, the sectors are delimited in a different way:

a and δ_2 delimit a white sector, then δ_2 and the ray e delimit the black sector and, again, we have a white sector delimited by e and b. These rays are used for the tree rooted at C. Similar rays, a_1, b_1 and e_1 are used for the tree rooted at C_1: as can be seen on the figure, the tree contains the one defined from C. Note that e_1 is the continuation of b. As in the case with the left-hand side picture, this picture also defines a bi-infinite word as the limit of w_n.

Note that, in both case, u_n tends to a limit which is infinite on one side only: this can be seen by the fact that the black sector is always delimited by δ_1 or δ_2 and these lines are fixed. The infinite limit is finite to the left in the case of (G_1), it is finite to the right in the case of (G_2).

In both cases, say that δ_1 and δ_2 are **separators**: δ_1 separates u_n from $w_n w_n$ for each n; δ_2 separates y_n from $x_n y_n$.

Figure 7 shows another way to separate $u_n w_n w_n$ or $y_n x_n y_n$. In Figure 6, the separation is between u_n and w_n in the first word, and in the second word it is between the separation the first y_n and x_n. In Figure 7, the separation in $u_n w_n w_n$ is between the two w_n's and it is in between x_n and the second y_n in $y_n x_n y_n$. These differences can be denoted as follows, in self-explaining notation:

$$\begin{array}{cc} u_n \bullet w_n w_n & u_n w_n \bullet w_n \\ y_n \bullet x_n y_n & y_n x_n \bullet y_n \end{array} \qquad (L_1)$$

Figures 8 and 9 illustrate two similar constructions leading to infinite limits for w_n and y_n which are infinite on one side only. The rays a, b and e play a somewhat different role with respect to the lines δ_1 or δ_2, compared to what we have seen in Figures 6 and 7. In Figure 8, the lines δ_1 and δ_2 define the leftmost side of the embedded trees. The rightmost side is b. The strip is delimited by δ_1 and a in the left-hand side picture. It is delimited by a and e in the right-hand side one. In both pictures, the rightmost sector is delimited between e and b. The lines a_1, b_1 and e_1 play similar roles with respect to the tree rooted at C_1. Note that in the left-hand side picture, the rays a_1 and e are supported by the same Euclidean line, while this is the case, in the right-hand side picture, for a_1 and b. From the picture, it is clear that this time the limit of w_n is infinite to the right and that it is also the case for y_n. In the case of (G_1), the limit of u_n is also infinite to the right only. In the case of (G_2), the limit of x_n is also infinite to the right. However, note that u_n and w_n have the same limit in this case, thanks to formula (E_1).

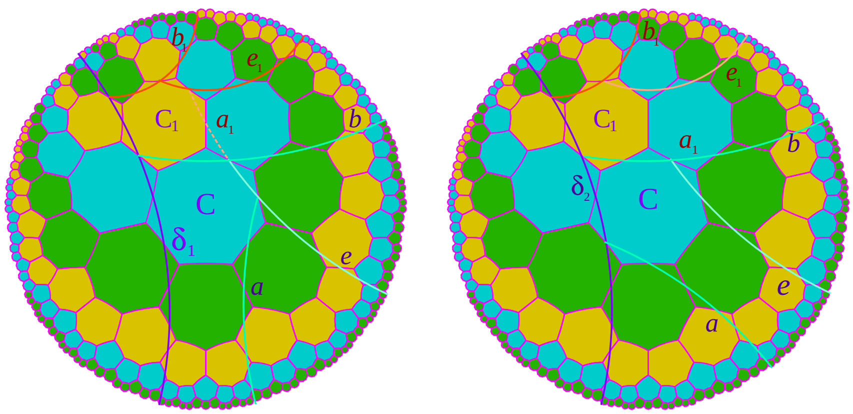

Figure 8 *Heptagrid: construction of one-sided infinite words, infinite to the right.*
Left-hand side: an infinite word associated with the grammar (G_1)*, Right-hand side: an infinite word associated with the grammar* (G_2)

$$\begin{array}{ll} \bullet u_n w_n w_n & \qquad u_n w_n w_n \bullet \\ \bullet y_n x_n y_n & \qquad y_n x_n y_n \bullet \end{array} \qquad (L_2)$$

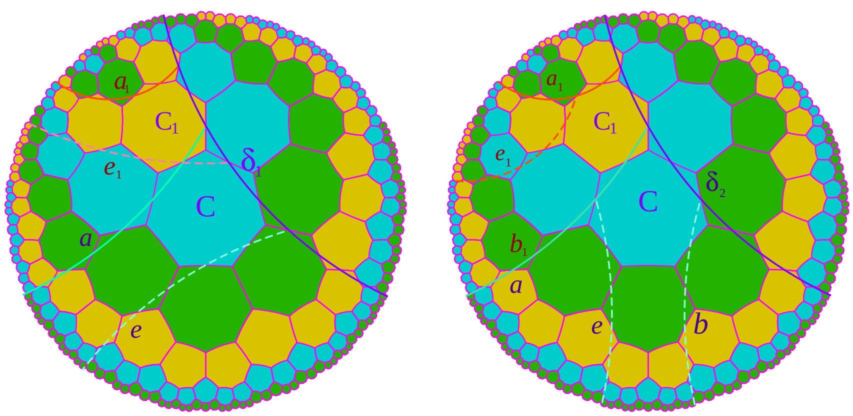

Figure 9 *Heptagrid: construction of one-sided infinite words, infintie to the left.*
Left-hand side: an infinite word associated with the grammar (G_1)*, Right-hand side: an infinite word associated with the grammar* (G_2)

In Figure 9, δ_1 and δ_2 delimit the rightmost side of the embedded trees. In the left-hand side picture, a is the leftmost side of the sector and the strip lies between a and e. Note that in this picture, the delimitation between the two regions giving rise to both copies of w_n is not mentioned. These copies lie between e and b. In the right-hand side picture, the strip is delimited by e and b. The leftmost region associated to x_n lies between a and e; the rightmost region also associated to x_n lies between b and δ_2. In both pictures, a_1, b_1 and e_1 play with respect to C_1 the same role as a, b and e respectively with respect to C. The corresponding words give rise to the second column in formula (L_2).

3 Thanks to Grossone: a more precise picture

As mentioned in Sub-section 1.2, the new numeral system allows us to look more precisely at what classical mathematics calls infinity. In [13], the author already applied this system to measure the number of tiles in the pentagrid and in the heptagrid, as well as in other tilings of the hyperbolic plane, and the interesting point was that this number depends on the way with which we look at the tiling. This is not surprising, according to Postulate 2.

Generalizing the fundamental equation of the Fibonacci sequence to any natural number, we proved in [13] that in a Fibonacci tree of height ν, where ν is any number of the numeral system, the number of leaves of the tree is $f_{2\nu+1}$. It is important here to note that we have no notion of limit: for each height ν we can write, we also can write $f_{2\nu+1}$, provided that this last number does not exceed the current limit of what can be written in the numeral system. Note that this restriction is due to Postulate 1.

Accordingly, let us pay a new visit to the construction of the infinite Fibonacci words defined in Sub-section 2.2. This time, with the new numeral system, it is plain that if ν is infinite, the words $u_\nu w_\nu w_\nu$ and $y_\nu x_\nu u_\nu$ are also infinite. We also can use the representation of these words in the hyperbolic plane as it is indicated in Sub-section 2.2. This time, we consider the sequence of nodes C_k with $k \in [0..\nu]$ and we look at the Fibonacci tree of height ν. The infinite words are the words obtained by traversing the leaves of the tree. We also have two kinds of such words due to the separation lines introduced in the discussion. Remember that these lines are the support of the sequence of the C_k's. We have the same difference between one-sided infinite words and bi-infinite words. But this time, we can measure the length of the words in all cases and, in the case of the bi-infinite words, we can measure the length of each infinite part.

Indeed:

Theorem 1 *Consider the infinite words defined in Sub-section* 2.2*. They give rise to infinite words whose length is given in Table* 1*, each case being characterized by the decomposition noted by the formulas* (L_1) *and* (L_2)*. In this table, when the left- or right-hand side part of a bi-infinite word is equal in length to a part of another bi-infinite word, the words themselves are not equal. They are also not equal to the reverse image of the other word.*

Proof. Let β_ν be the number of leaves of the tree $\mathcal{T}_\nu$ of height ν constructed with the rules defining the Fibonacci tree but starting from a black node. It is plain that $\beta_{\nu+1} = f_{2\nu+3} - f_{2\nu+1}$ as we remove a Fibonacci tree of height ν from $\mathcal{T}_{\nu+1}$, according to the second equality in formula (E_1). Accordingly, $\beta_{\nu+1} = f_{2\nu+2}$.

For the one-sided infinite words, the assertion of the theorem comes from the fact that in both cases the length is $f_{2\nu} + 2f_{2\nu+1}$. This can be rewritten as:

$$f_{2\nu} + 2f_{2\nu+1} = f_{2\nu} + f_{2\nu+1} + f_{2\nu+1} = f_{2\nu+2} + f_{2\nu+1} = f_{2\nu+3}.$$

For the bi-infinite words, the length of $x_\nu y_\nu$ is $\beta_\nu = f_{2\nu}$, which gives us the assertion of the theorem.

For the difference, we already know that $w_n \neq y_n$ and that $u_n \neq x_n$ although in each comparison, the words have the same length. The other possible comparisons are with $u_\nu w_\nu$, $y_\nu x_\nu$ and their reverse images. For $u_\nu w_\nu$ and $y_\nu x_\nu$, $u_\nu w_\nu$ starts with a B and ends with a W. Now, $y_\nu x_\nu$ starts and ends with a W. Accordingly, $u_\nu w_\nu$ is not equal to $y_\nu x_\nu$. Moreover, it is not equal neither to the reverse of $y_\nu x_\nu$ nor of itself. For $y_\nu x_\nu$, x_ν and y_ν, we have just to prove that they are not palindroms. Now, it is clear that if we start from the leftmost letter of x_ν, y_ν or $y_\nu x_\nu$, the fourth letter is B and if we start from the rightmost letter of x_ν, the fourth letter is W, this easily comes from (E_2). Note that this argument requires that $\nu > 2$, which is of course the case for an infinite number which is not necessarily bounded by **1**. $\blacksquare$

Table 1 *Table of the lengths of parts of infinite Fibonacci words depending on their overall patterns, ν being an infinite number.*

`pattern`	`length`	`left`	`right`
$\bullet u_\nu w\nu w_\nu$	$f_{2\nu+3}$	0	$f_{2\nu+3}$
$\bullet y_\nu x\nu y_\nu$	$f_{2\nu+3}$	0	$f_{2\nu+3}$
$u_\nu \bullet w\nu w_\nu$	$f_{2\nu+3}$	$f_{2\nu}$	$2f_{2\nu+1}$
$y_\nu \bullet x\nu y_\nu$	$f_{2\nu+3}$	$f_{2\nu+1}$	$f_{2\nu+2}$
$u_\nu w\nu \bullet w_\nu$	$f_{2\nu+3}$	$f_{2\nu+2}$	$f_{2\nu+1}$
$y_\nu x\nu \bullet y_\nu$	$f_{2\nu+3}$	$f_{2\nu+2}$	$f_{2\nu+1}$
$u_\nu w\nu w_\nu \bullet$	$f_{2\nu+3}$	$f_{2\nu+3}$	0
$y_\nu x\nu y_\nu \bullet$	$f_{2\nu+3}$	$f_{2\nu+3}$	0

As we can see, we have several families of infinite words. We have no more limits but different numbers with different infinite lengths which we can precisely measure. This also allows us to define new families which could not be distinguished in the classical approach. Consider the bi-infinite word $u_\nu \bullet w_\nu w_\nu$. Let us consider two infinite integers, ν and λ with $\lambda < f_{2\nu+1}$. Then, we can fix a word φ_λ by $\varphi_\lambda = u_\nu w_\nu^1 w_\nu^2 w_\nu$ with the conditions $w_\nu^1 \bullet w_\nu^2 w_\nu = w_\nu$ and $|w_\nu^1| = f_{2\nu+1} - \lambda$, where $|m|$ is the length of the word m. Similarly, with the condition $\lambda < f_{2\nu}$, we can define a word $\varphi_{-\lambda}$ by $\varphi_{-\lambda} = u_\nu^1 \bullet u_\nu^2 w_\nu w_\nu$, with the conditions $u_\nu^1 \bullet u_\nu^2 = u_\nu$ and $|u_\nu^1| = f_{2\nu} - \lambda$. In φ_λ, the length of the left-hand part is $f_{2\nu+2} - \lambda$ and the length of the right-hand side part is $f_{2\nu+1} + \lambda$. Similarly, in $\varphi_{-\lambda}$, the length of the left-hand side part is $f_{2\nu} - \lambda$ and that of the right-hand side part is $2f_{2\nu+1} + \lambda$. Similar families can be defined for $u_\nu w_\nu \bullet w_\nu$, $y_\nu \bullet x_\nu y_\nu$ and $y_\nu x_\nu \bullet y_\nu$ with the appropriate conditions on λ.

Such families can be simply defined in geometrical terms when $\lambda = f_{2\eta+1}$

with η being an infinite integer. Indeed: on δ_1, we take a tile T at level η and using T, we draw a mid-point line ℓ such that ℓ and δ_1 delimit a Fibonacci tree rooted at T. It is possible to draw ℓ in any side of δ_1. Note that we can also define a position of a mid-point line ℓ defining with δ_1 a Fibonacci tree rooted at a black node. However, to locate this tile requires the use of the new numeral system. Now, this new system also allows us to define these families even when λ is not a Fibonacci number, even in a very simple fashion.

4 Conclusion

Accordingly, the new system opens the way to the study of new questions we have indicated. We shall make more precise two of them.

The first question is whether it is possible to describe gemetrically the process described in the construction of a φ_λ or $\varphi_{-\lambda}$ family. We have seen that the construction is possible when λ has the form f_η. This can be done if we can represent any λ as a sum of Fibonacci numbers. Now, there may be infinitely many numbers in the sum. This can be seen from formulas (C_f) and (C_g) from which we derive:

$$f_{2\nu+1} - 1 = \sum_{i=1}^{\nu} f_{2i} \qquad\qquad f_{2\nu} - 1 = \sum_{i=1}^{\nu-1} f_{2i+1}, \qquad\qquad (C_\infty)$$

where ν may be an infinite integer represented in the new numeral system. In the case of an irreductibly infinite decompostion, we cannot perform a geometrical construction which requires finitely many actions according to Postulate 1.

The particular case of (C_∞) can also be dealt with simply but we can probably derive more complex cases from the very formula (C_∞). And so we have a second question about the decomposition of infinite numbers into a sum of Fibonacci numbers, also infinite and, possibly, with infinitely many terms. Can the new system tell us something about such a decompsition which may be not unique as (C_∞) itself witnesses?

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