ISSN 0234-0453 • INFORMATICS AND EDUCATION • 2022 • Volume 37 No 1

DOI: 10.32517/0234-0453-2022-37-1-79-86

A COMPUTATIONAL POINT OF VIEW ON TEACHING DERIVATIVES

F. Mazzia[1] ✉

[1] ***University of Bari Aldo Moro, Bari, Italy***

✉ *francesca.mazzia@uniba.it*

Abstract

The strong connection between mathematics and informatics highlights the important role of scientific computing in many applications. Unfortunately, mathematics is traditionally taught without investigating possible connections between abstract problem solving and the use of algorithms capable of being implemented on a computer. Since mathematical theory and computing practice are taught separately, many students fail to appreciate the utility of mathematics. In this paper, we briefly explain how a typical lecture on obtaining derivatives in differential calculus can benefit from examples implemented in high-level languages like MATLAB, Python or R. Such examples can help to guide the students to a better understanding of the theoretical concepts and limits of finite precision floating-point arithmetic. We argue that, while the historical findings of Leibniz and Newton are good starting points for introducing finite difference approximation methods, this does not preclude new approaches to numerically computing derivatives using Infinite Computer Arithmetic.

Keywords: derivatives, floating-point arithmetic, Infinite Computer Arithmetic

For citation:

Mazzia F. A computational point of view on teaching derivatives. *Informatics and Education.* 2022; 37(1):79–86. DOI: 10.32517/0234-0453-2022-37-1-79-86

ВЫЧИСЛИТЕЛЬНЫЙ ВЗГЛЯД НА ОБУЧЕНИЕ ПРОИЗВОДНЫМ

Ф. Маццна[1] ✉

[1] ***Университет имени Альдо Моро в Бари, г. Бари, Италия***

✉ *francesca.mazzia@uniba.it*

Аннотация

Тесная связь между математикой и информатикой подчеркивает важную роль научных вычислений во многих приложениях. К сожалению, математика традиционно преподается без изучения возможных связей между решением абстрактных задач и использованием алгоритмов, которые можно реализовать на компьютере. Поскольку математическая теория и вычислительная практика преподаются отдельно друг от друга, многие студенты не понимают полезности математики. В этой статье мы кратко объясняем, как на типичной лекции о получении производных в дифференциальном исчислении могут быть использованы примеры, реализованные на языках высокого уровня, таких как MATLAB, Python или R. Такие примеры могут помочь студентам лучше понять теоретические концепции и пределы арифметики с плавающей запятой конечной точности. Мы утверждаем, что, хотя исторические открытия Лейбница и Ньютона являются хорошей отправной точкой для введения методов аппроксимации конечных разностей, это не исключает новых подходов к численному вычислению производных с использованием бесконечной компьютерной арифметики.

Ключевые слова: производные, арифметика с плавающей запятой, длинная арифметика.

Для цитирования:

Маццна Ф. Вычислительный взгляд на обучение производным. *Информатика и образование.* 2022;37(1):79–86. (На англ.) DOI: 10.32517/0234-0453-2022-37-1-79-86

1. Introduction

As it is explained in an excellent way in [1] mathematic is an indispensable tool, especially in the field of computer engineering. It is used for analysis of simulation, for optimal control techniques and it is extremely important in Artificial Intelligence and Data Mining.

It is a well-known problem that students often consider mathematics not useful. This can be also due to the omission of any connection between mathematics and the possibility of solving problems using an algorithm implemented numerically on a computer.

This happens both in school and in the first years of higher degree educations. Our experience with students in the second year of the bachelor's degree in computer science is that most of them think that they can be excellent computer scientist without studying mathematics. So, we make examples to try to change their mind. One of the first topic we use is the concept of derivatives, one of the fundamental concepts of mathematics, useful to understand a lot of applicative tools like Taylor polynomials, differential equations, Fourier transform and, finally, Neural Networks.

In the following sections we briefly explain how a typical lecture related to derivatives can use examples implemented by the students in a problem-solving environment, like MATLAB [2], Python [3] or R [4], to guide the students to better understand the theoretical concepts and the limits of finite precision floating-point arithmetic. The historical results of Fermat, Descart, Leibniz and Newton are good starting points for introducing finite difference approximations methods. After implementing the incremental ratio on a computer, we show pictures and discuss the results. Finally, we show how, using the Infinite Computer Arithmetic introduced in [5] to implement the incremental ratio, some of the limitations of the floating-point arithmetic can be overcome. We also show how the weakness in the theory of the original methods to compute maxima and tangents can be handled with the acquired competence, since the Infinite Computer Arithmetic allows to deal with Infinite and Infinitesimal numbers in a simpler way [5, 6].

2. Fermat, Descart, Newton, and Leibniz

«Historically speaking, there were four steps in the development of today's concept of the derivative, which I list here in chronological order. The derivative was first used; it was then discovered; it was then explored and developed; and it was finally defined». This is part of the introduction of the article [7], a good historical guide through the steps of the concept of derivatives when calculus was not yet developed. Knowing the history can inspire students: guidelines on how to develop and teach historical material can be found in [8].

Historically, derivatives are associated to Pierre Fermat's methods of finding maxima and minima. A good example is the same simple problem solved by Pierre Fermat in 1630 described in [7]. The problem is to find the subdivision of a segments so that the product of the length of the two parts is maximum. If L is the length of the segment and x the length of one of its parts, the product is a polynomial given by $p(x) = x(L-x) = xL - x^2$, where L is given and x is our unknown.

At that time, derivatives were not known. The only information Fermat had to solve this problem was the technique used by Pappus of Alexandria that lived in the 320 AD and collected in his writing the most important works done in ancient Greek, allowing them to survive. Many mathematicians were influenced by this collection, not only Fermat, but also René Descartes and Isaac Newton. The information taken by the work of ancient Greeks was that a problem that has, in general, two solutions has only one solution in case of maximum.

Fermat defines the second solution $x+h$, and $p(x+h) = (x+h)(L-x-h) = xL - x^2 - 2xh + hL - h^2$.

If the solution is one such that: $p(x) = p(x+h)$, that is: $xL - x^2 = xL - x^2 - 2xh + hL - h^2$, we have that $0 = -2xh + hL - h^2$.

Now Fermat assert that we can suppress h and find $x = L/2$.

The student can now be interested in showing the relation with the modern concept of derivative. Looking at the method proposed by Fermat we can show that it is really performing the difference $p(x+h) - p(x)$, dividing by h and suppressing h, that for many examples, corresponds to the limit of the incremental ratio.

Fermat theory was not accepted, because it was not rigorous. The division by h is not allowed. However, this method historically was extremely useful, and many mathematicians used it. The main weakness of the method is the division by a number h that is later considered.

The incremental ratio enters directly in use by the seventeenth-century mathematicians, like Fermat and Descart to solve the problem to find the tangent of a curve represented by the equation $y = f(x)$. The method starts by computing the slope of the secant, which is exactly the incremental ratio. Figure 1 suggests that when h vanishes the slope of the secant gives the slope of the tangent. In this case we have the same weakness of the maximum problem of Fermat: setting $h = 0$ after the division by h.

The concept of derivative and of the modern calculus has been introduced by Isaac Newton and Gottfried Leibniz independently and the notations of derivative used nowadays are both the one by Newton ($\dot{x}$) and the one by Leibniz ($\frac{dy}{dx}$). The concept of limit was not yet introduced and the concept of infinitesimals and indefinitely small quantities left a lot of unanswered questions, that were hardly resolved with practical consideration. Nevertheless, with this basis, many important theories have been developed.

These two simple historical examples can help students learning how the concept of derivative was born, even if they have not clearly developed competences on the limit concept. The definition of derivative as limit of the incremental ratio, indeed, was introduced by Cauchy in 1823, who defined the concept of limit, that was later analyzed by many authors. The limit definition we use today, instead, is attributed to Weierstrass in 1850.

Observe that even if the incremental ratio was well known in the seventeenth-century we must wait until the nineteenth-century to have the definition of the derivative as limit.

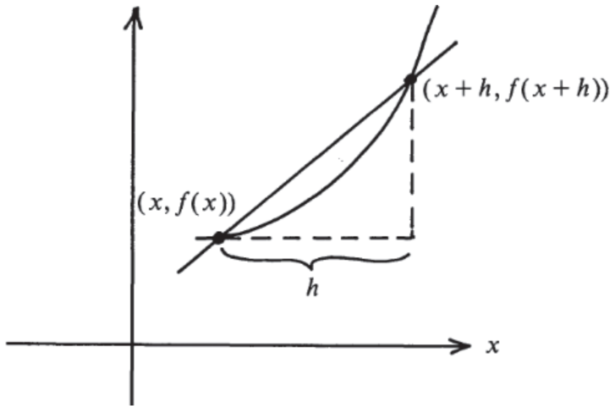

Fig. 1. Slope of the secant [7]

3. Derivatives and incremental ratio

Derivatives are usually explained starting from the problem of finding the tangent of a curve, defining the incremental ratio of a function, and using the limit concept giving the well-known definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

if the limit exists and it is well defined. A list of fundamental derivatives is then derived studying the limit of the incremental ratio. The symbolic manipulation of the expressions is necessary to compute the limit to reach a representation of the derivative using symbols.

The importance of the use of technology and in particular computer algebra systems in preforming graphical representation and procedures is emphasized in many papers, as, for example, [9, 10]. The discrete approach is also considered essential to understand the limit concept [11]. The importance of the *concept content*, the *concept scope*, and the *concept network* in teaching mathematics and, in particular, limits and derivatives, is well described in [12].

In my opinion, the use of a high-level computer language and a problem-solving environment to teach algorithms to compute the derivative could help to understand not only the mathematical concept, but also how the computer works with real numbers. This aspect is usually not taken into consideration.

Having explained the derivative using the standard way, we can support the symbolic computation of the limit with the graphical and algorithmic representation. This will be also useful to understand the behavior of computers, that cannot work with the infinite set of real numbers, but only with a finite subset.

We start by implementing the incremental ratio of a function for which the derivative is known. We can choose polynomials as first examples and after we can move on with trigonometric functions.

Figure 3 and Figure 4 show what happens if we use different increment values on the function $f(x) = x^2$, plotted in Figure 2.

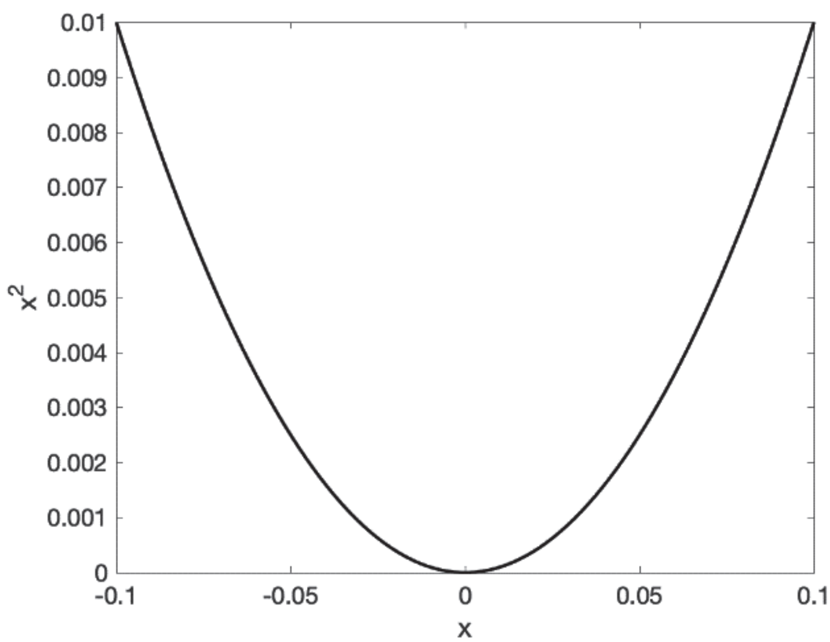

Fig. 2. Function f(x) in the interval [−0.1, 0.1]

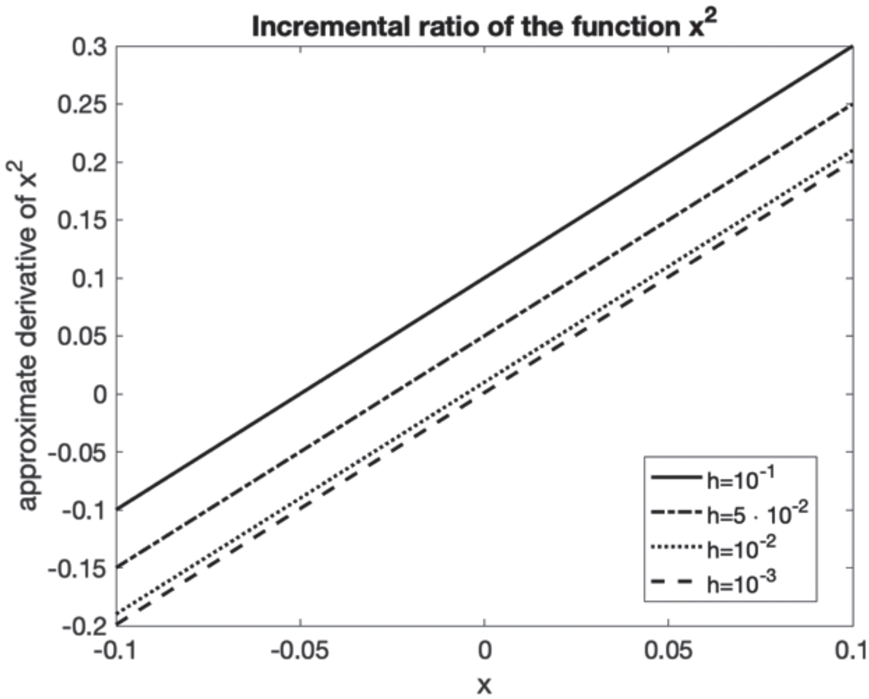

Fig. 3. Picture of the incremental ratio using different values of h for the function x^2

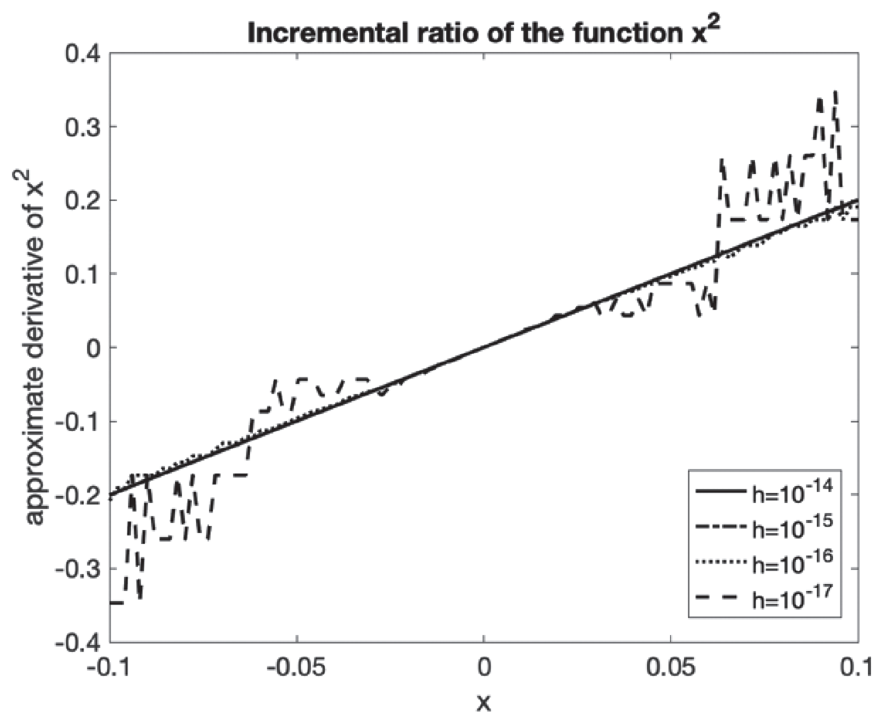

Fig. 4. Picture of the incremental ratio using very small different values of h for the function x^2

The plot of

$$f_h(x) = \frac{f(x+h) - f(x)}{h}$$

in Figure 3 is done by using $h = 0.1$, $h = 0.05$, $h = 10^{-2}$, $h = 10^{-3}$. The used increments clearly show the dependence of the incremental ratio on h. With the smallest h we find a function that cannot be distinguished from a graphical point of view. We ask the student to compute the better approximation that they can. It is sure that they will try to compute the ratio using very small values of h. Figure 4 shows that, using $h = 10^{-14}$, $h = 10^{-15}$, $h = 10^{-16}$, $h = 10^{-17}$, the derivative is not approximated correctly and that decreasing h the error increases. Now we can go on and symbolically compute the function

$$s_h(x) = \frac{(x+h)^2 - x^2}{h} = 2x + h,$$

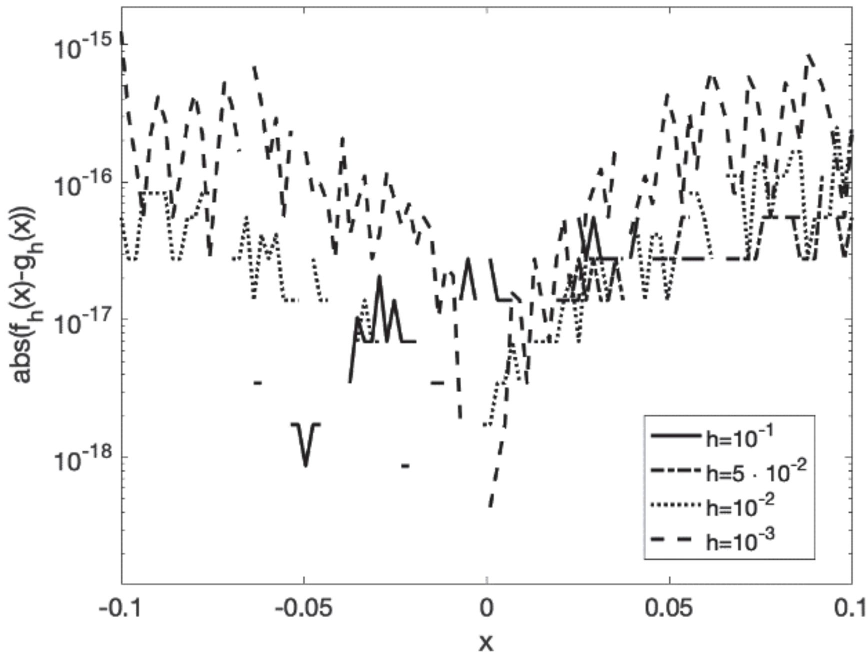

Fig. 5. Absolute error of the numerically computed incremental ratio and of the symbolically computed one

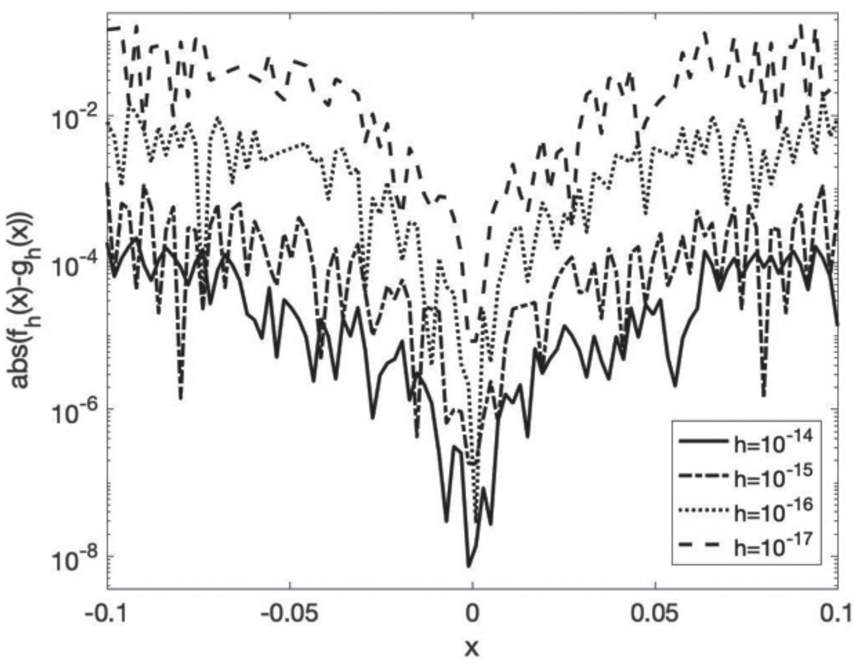

Fig. 6 Absolute error of the numerically computed incremental ratio and of the symbolically computed one using small values of h

and check if the results change by implementing this function. In Figure 5 and Figure 6 we now plot the absolute error $(abs(f_h(x) - s_h(x)))$ using the same values of h. Analyzing the plots, it is possible to see that in the first case the incremental ratio computed using the computer and the one computed using the symbolic computation have for each h a value that oscillates in the interval $[0, 10^{-15}]$. This output is useful to observe that the results are not exactly the same, but the error, i. e. the difference, is small and it is related to the arithmetic used by the computer that is based on a finite subset of the infinite real numbers.

Figure 6 shows the same error but using smaller values of h: in this case the error is higher and grows when h becomes smaller. The symbolic computation this time gives the correct result, even if theoretically they should give the same results. This is confirmed by the interesting plot of the maximum absolute error of the approximate derivative with respect to the true one, computed changing h in the interval $[0, 0.1]$. Now we fix $x = 0.05$ and we compute $abs(f'(x) - f_h(x))$.

The picture in Figure 7 shows that the error decreases linearly changing h, it reaches a minimum value and after it starts to increase in a random way.

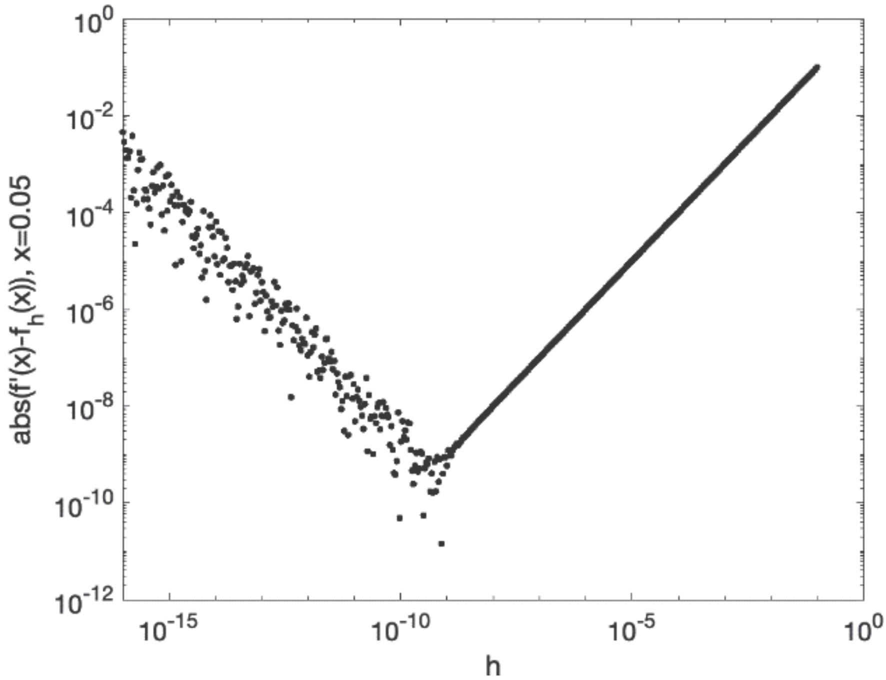

Fig. 7. Absolute error with respect to the exact derivative changing h

This is a simple and good example to show that, working with computer is not the same as working with symbols. The graphical representation of the derivative and its approximation will help the student to better understand the incremental ratio behavior and its limit, and could be used as a first example to show that the arithmetic used by computers is different from the standard one. Many other examples can be performed changing the function, in particular this technique could be used to compute approximations of derivatives for which it is difficult or, in some cases, impossible to compute symbolically the exact one (for example, black-box functions).

Nowadays symbolic calculus for the approximation of derivatives is very well established and software is available to the students for this purpose [13, 14]. The use of numerical approximation is, however, necessary in many real-world applications. A practical introduction to the floating-point arithmetic experimenting with the incremental ratio gives the students a first important idea of some of the main problems of scientific computing [15].

4. Infinite Computer Arithmetic

The incremental ratio is at the same time a very good mathematical problem that could be used in class for a practical application of a more recent kind of arithmetic, the Arithmetic of Infinity, that allows to work with infinitely large and infinitely small numbers. This arithmetic, if implemented on a computer, allows to solve the problems of using small h in the incremental ratio working with both floating-points numbers and the numeral system with base '*gross-one*' denoted by the symbol ①, that represents the number of elements of the set of natural numbers. This numeral system has been introduced by Yaroslav Sergeyev in 2003. A comprehensive description can be found in his book "Arithmetic of Infinity" [5]. It should be stressed that one of the main distinctions of this methodology with respect to the well-known non-standard analysis [16] consists in its pronounced numerical character allowing one to work with floating-point numbers on the Infinity Computer whereas non-standard analysis is symbolic [17].

This Arithmetic has already been used in secondary schools; the reader is refereed to [6] for useful didactic material. Here we introduce only what is necessary for our experiments for the computation of derivatives. Many scientific works have been published on this subject: the reader is referred to [18, 19] for an advanced theoretical description.

In the Arithmetic of Infinity, a number is called *grossnumber* and is expressed using the base ① as a linear combination of power of the form

$$C = c_{p_m}①^{p_m} + \ldots + c_{p_1}①^{p_1} + c_{p_0}①^{p_0} + c_{p_{-1}}①^{p_{-1}} + \ldots + c_{p_{-l}}①^{p_{-l}},$$

where all $c_i \neq 0$ are numerals belongings to a traditional numeral system and are called *grossdigits*, while the numerals p_i, that can be grossnumbers, are sorted in decreasing orders with $p_m > \ldots > p_0 = 0 > \ldots > p_{-l}$, and are called *grosspowers*. The finite part of a grossnumber is the grossidigit associated to $p_0 = 0$, the infinite part is represented by the grossidigit associated to finite or infinite $p_m, \ldots, p_1$ and the infinitesimal part by the grossdigits associated to finite or infinite $p_{-1}, \ldots, p_{-l}$.

Examples of infinitesimal numbers are

$$C_1 = 4①^{-1} + 2.78①^{-5},$$

$$C_2 = 8.9①^{-2.2} + 3.2①^{-3}.$$

Examples of infinite numbers are

$$C_3 = 6①^3 + 3.28①^1 + 0.5①^0,$$

$$C_4 = 1.2①^{3.5} + 7.3①^{1.45} + 1.26①^{-3}.$$

For this numeral system it is possible to use the positional notation and to define the arithmetic operations. A comprehensive description is given in [5], while it is possible to find a survey of many results and scientific works in [20].

Now let us consider the incremental ratio. Since using small h the computer does not allow us to have good results, we perform the arithmetic operations using infinitesimal increments. A good experiment is to show what happens when we choose $h = ①^{-1}$. The incremental ratio for our test function $f(x) = x^2$ is

$$g_{①^{-1}}(x) = \frac{(x + ①^{-1})^2 - x^2}{①^{-1}}.$$

Now we can compute the arithmetic operation and the result is

$$g_{①^{-1}}(x) = 2x①^0 + ①^{-1}.$$

If x is a real number and the derivative exists, the finite part of $g_{①^{-1}}(x)$ is equal to the exact derivative.

The use of the Infinite Arithmetic allows us to define the derivative using the incremental ratio with an infinitesimal step, without needing the definition of limit. Many examples related to the computation of limits using the Arithmetic of Infinity can be found in [5].

Note that routines based on this arithmetic are available on the computer, and the computation is different from the symbolic calculation [19]. To learn more about the potentiality of the Infinity Computer Arithmetic, it is possible to perform the same steps of Fermat using $h = ①^{-1}$. What we obtain is $h = \frac{L}{2} - \frac{①^{-1}}{2}$, and we can consider as result the finite part of the sum. No contradiction appears anymore.

Naturally, for students in the secondary school it is possible to compute by hand only simpler arithmetic operations. Using the software that computes the derivatives in the Infinity Arithmetic, instead, a graphical representation of derivatives that symbolic calculus handle with difficulty can be made without problems.

5. Conclusion

Starting to explain tangents using secants and experimenting with computation in floating-point arithmetic are helpful steps to understand derivatives and how computers work. The second step is to explain derivatives changing the way the computation is done and using the Infinity Computer Arithmetic. At the end, following the historical steps, we can explain the standard limit definition, that we need if we cannot work with the Infinity Computer Arithmetic. In my opinion, this is an attractive approach that introduces students to general aspects associated to the solution of mathematical problems using the computer.

Acknowledgements

The author thanks Felice Iavernaro, Antonella Falini and Giuliana Galati for the support and useful discussions during the preparation of the paper and the anonymous reviewers for their useful suggestions. The author is member of the INdAM Research group GNCS.

References

1. *Buttazzo G.* L'importanza della Matematica nell'Era Tecnologica. *L'educazione Matematica, Anno XXXVII-Serie X*, 2016;6(2):9–26. Available at: https://retis.santannapisa.it/~giorgio/paps/2016/EduMat16.pdf

2. Matlab, Mathworks, 2021. Available at: https://it.mathworks.com/products/matlab.html

3. Python, Python Software Foundation, 2021. Available at: https://www.python.org/

4. R, The R foundation, 2021. Available at: https://www.r-project.org/

5. *Sergeyev Y. D.* Arithmetic of Infinity. *Edizione Orizzonti Meridionali*. 2003. 112 p. DOI: 10.1007/s10898-005-6469-6

6. *Rizza D.* Numerical methods for infinite decision-making processes. *Int J Unconvent Comput*. 2019;14(2):139–158. Available at: https://ueaeprints.uea.ac.uk/id/eprint/69574/

7. *Grabiner J. V.* The changing concept of change: The derivative from Fermat to Weiestrass. *Mathematics Magazine*. 1983;56(4):195–206. Available at: https://scholarship.claremont.edu/cgi/viewcontent.cgi?article=1118&context=pitzer_fac_pub

8. *Su Y.-W.* Research and practice in developing historical materials in teaching calculus. *International Journal of Information and Education Technology*, 2019;9(3):189–195. Available at: http://www.ijiet.org/vol9/1197-SJ016.pdf

9. *Herbert S.* Challenging the traditional sequence of teaching introductory calculus. *Computers in the Schools: Interdisciplinary Journal of Practice, Theory, and Applied Research*. 2013;30(1-2):172–190. Available at: https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.949.8228&rep=rep1&type=pdf

10. *Zhang J.* How to use computer technology to break through the difficulties in higher mathematics teaching. *IOP Conference Series: Materials Science and Engineering PAPER*. 2020;750:012039. DOI: 10.1088/1757-899X/750/1/012039

11. *Weigand H. G.* A discrete approach to the concept of derivative. *ZDM Mathematics Education*. 2014;46:603–619. DOI: 10.1007/S11858-014-0595-X

12. *Weigand H.-G., Greefrath G., Oldenburg R., Siller H.-S., Ulm V.* Aspects and "Grundvorstellungen" of the concepts of derivative and integral subject matter-related didactical perspectives of concept formation. *Journal für Mathematik-Didaktik*. 2016;37:99–129. DOI: 10.1007/s13138-016-0100-x

13. WolframAlpha computational intelligence. Available at: https://www.wolframalpha.com

14. SageMath. Available at: https://www.sagemath.org

15. *Ascher U., Greif C.* A first course in numerical methods. *Computational Science & Engineering. SIAM*. 2011. 552 p. DOI: 10.1137/1.9780898719987

16. *Robinson A.* Non-standard Analysis, Princeton: Princeton Univ. Press, 1996. 308 p.

17. *Sergeyev Y. D.* Independence of the grossone-based infinity methodology from non-standard analysis and comments upon logical fallacies in some texts asserting the opposite.

Foundations of Science. 2019;24(1):153–170. DOI: 10.1007/s10699-018-9566-y

18. *Sergeyev Y. D.* Higher order numerical differentiation on the Infinity Computer. *Optimization Letters*. 2011;5(4):576–585. DOI: 10.1007/s11590-010-0221-y

19. *Iavernaro F., Mazzia F., Mukhametzhanov M., Sergeyev Y.* Computation of higher order Lie derivatives on the Infinity Computer. *Journal of Computational and Applied Mathematics*. 2021;383:1131135. DOI: 10.1016/j.cam.2020.113135

20. *Sergeyev Y. D.* Numerical infinities and infinitesimals: Methodology, applications and repercussions on two Hilbert problems. *EMS Surveys in Mathematical Science*. 2017;4:219–320. DOI: 10.4171/EMSS/4-2-3

Information about the author

Francesca Mazzia, Full Professor of Numerical Analysis at the Department of Information Technology, University of Bari Aldo Moro, Bari, Italy; *ORCID:* http://orcid.org/0000-0003-1072-9578; *e-mail:* francesca.mazzia@uniba.it

Информация об авторе

Мацциа Франческа, профессор департамента информатики, Университет имени Альдо Моро в Бари, г. Бари, Италия; *ORCID:* http://orcid.org/0000-0003-1072-9578; *e-mail:* francesca.mazzia@uniba.it

Поступила в редакцию / Received: 01.12.2021.
Поступила после рецензирования / Revised: 14.01.2022.
Принята к печати / Accepted: 15.01.2022.