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CONTENTS

The Effect of *Arithmetic of Infinity* Methodology on Students' Beliefs of Infinity

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Abstract

This study investigates the effect of a new methodology of the mathematical infinity on students' beliefs of it. A group of 25 students (grade 12) participated in the study considering traditional views on infinity in mathematics and then recently introduced to "*Arithmetic of Infinity*" methodology proposed by Yaroslav D. Sergeyev. The students were subject to an intervention presenting this methodology which is new to them and different from the classical one. A pre- & post questionnaires were administrated to check its effect. Results showed that the majority of students had positive beliefs of infinity after being exposed to this methodology.

Keywords: Arithmetic of Infinity, beliefs, grossone, infinity, students' perception of infinity.

Introduction

Infinity is a fantastic concept that has had a long history of struggle in the human mind. It has puzzled minds of great mathematicians and philosophers. Cornu (1991) considered the notion of the infinitely large and the infinitely small as one of the major epistemological obstacles of the past (in Moru, 2006). It was not until the mid-twentieth century that infinity was put on rigorous mathematical basis after the Cantorian set theory and the theory of non-standard analysis. Nevertheless, from a pedagogical perspective these obstacles still persist.

The concept of infinity is no doubt of great importance in mathematics as it is integrated in many branches of it. But due to its abstract nature, students usually have hard times dealing with it. Previous research asserts that most students, pre-service and in-service teachers hold misconceptions of infinity (Aztekin, Arikan & Sriraman, 2010; Schwarzenberger & Tall, 1978; Tall, 1980; Weller et al., 2009; Kattou et al., 2010).

In a previous study done in Lebanon regarding students' and teachers' conceptions of infinity, it was noticed that students hold numerous misconceptions of infinity and that they memorize and apply properties related to infinity without being convinced. Results showed that even teachers don't have a coherent conception of infinity (Nasr, 2015).

Regarding comparison of infinite sets, students cannot swallow the fact that a part and a whole could be of the same size, even if we show them the one-to-one correspondence. Many studies have investigated and analyzed students' approaches in comparing infinite sets (Singer & Voica, 2003; Singer & Voica, 2009; Kattou et al., 2010; Tirosh, 1999; Tsamir & Tirosh, 1999) and other studies were devoted to draw students' attention from using the part-whole method (which is intuitive and makes sense to students) and try to shift their thinking to using the one-to-one correspondence in comparing infinite sets (Tsamir & Tirosh, 1999; Singer & Voica, 2009).

Tall (1980) suggested that extrapolating measuring properties of numbers can lead to an intuition of different sizes of infinity. Hazzan (1999) suggested reducing the level of abstraction of certain concepts by making use of familiar procedures to tackle with unfamiliar situations. Similarly, Fischbein (2001)[1] found that when dealing with abstract or complex concepts, our reasoning tends to replace them by more accessible and familiar mental models. These *tacit models* (usually

[1] Professor Fischbein passed away in 1998. This paper that he had left was published in 2001 after being edited by Tommy Dreyfus.

figural) lead to contradictory interpretations. Dubinsky et al. (2001) proposed that interiorization of a process leads to an understanding of the potential infinity whereas, encapsulation of the process leads to an understanding of the actual infinity. Singer (2009) found that students tend to construct various structures when they are asked to find cardinality of an infinite set or cardinal equivalency of infinite sets. In another study, Singer & Voica (2008) explored children's primary and secondary perceptions[2] of infinity and found that these perceptions change from *processional* to *topological* across development. However, both the processional and topological perceptions co-exist and collaborate according to the given task in almost all ages. Kratka et al (2021) researched the development of students' conceptions of infinity in four combinations of views (distance & depth) and contexts (arithmetical & geometrical). They found out that the proportional representation of individual conceptions of infinity is highly dependent on context and view.

Tirosh (1999) found out that the majority of students use *iteration process method* when they are asked to determine finite/infinite nature of a given set. In the same study, Tirosh noted that the contradictions faced by students regarding infinite sets could serve in increasing their awareness of the importance of the formal definitions in mathematics. Tirosh & Tsamir (1996) studied the effect of representations of infinite sets on students' decisions of equivalency between given infinite sets. Results showed that the students mainly used one-to-one correspondence in numerical explicit and geometrical representations.

Despite all the attempts in literature to overcome conflictions with infinity, yet the struggle in students' minds seems never ending. For this reason, a new methodology, due to Yaroslav

[2] Primary perception is defined as: "an active and spontaneous process by which human beings organize and interpret sensory information, independently of any instruction."
Secondary perception: "is a filter of selection, interpretation and representation of information, which is created by successive experiences, generated inclusively by systematical educational interventions." (Singer & Voica, 2008)

Sergeyev[3] for dealing with infinity and in particular, comparing infinite sets was applied in this study to check its efficiency. Our hypothesis was that this new methodology provides an easy and practical way for students regarding the infinity concept.

This study raises the following questions:

- How would grade 12 students interact with the new methodology introduced to them? Will they be able to grasp its basics and perform computations with infinity easily and in a relatively short period of time?
- What are students' beliefs of the mathematical infinity? And would these beliefs be altered after being exposed to the new approach?

The Arithmetic of Infinity in Brief

The Arithmetic of Infinity was introduced in 2003 by Yaroslav Sergeyev. This arithmetic provides a new methodology of dealing with the infinite and infinitesimals. It is widely applied in several fields of mathematics: calculus, ordinary differential equations and optimization, set theory, fractals, Euclidean and hyperbolic geometry, probability, game theory, infinite series, cellular automata etc. (see Calude and Dumitrescu, 2020; D'Alotto, 2017; 2020; De Leone et al, 2020; Fiaschi and Cococcioni, 2018; Iavernaro et al, 2021; Rizza, 2018; Sergeyev et al., 2016; Sergeyev, 2009; 2010; 2013; 2016; 2017; 2021; Antoniotti, Caldarola, & Maiolo, 2020; Caldarola, Maiolo & Solferino, 2020). The non-contradictoriness of the Arithmetic of Infinity has been studied in several works (see for example, Lolli, 2015; Margenstern, 2011). The methodology had been also

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[3] Y. D. Sergeyev, DIMES, Università della Calabria, Via P. Bucci, Cubo 42-C, 87036 Rende (CS), Italy; Lobachevsky University, Nizhni Novgorod, Russia; and Institute of High Performance Computing and Networking of the National Research Council of Italy, Rende, Italy
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tested in some studies in pedagogy and had proved to be fruitful (Iannone et al., 2018; Antoniotti, Caldarola, d'Atri & Pellegrini, 2020; Ingarozza et al., 2020).

Sergeyev (2010; 2013; 2017; 2021) argues that the complications related to infinity are not a result of the abstract nature of infinity, but of our inability to distinguish between its different sizes. As a consequence, indeterminate forms arise and we find difficulties in dealing with infinity properly. He points out the importance of separation between the mathematical objects and the numeral[4] systems representing them. His argument shows that this new notion doesn't contradict that of Cantor's or non-standard analysis. Instead, it uses a sharper lens that provides more accurate insight of different infinities.

Sergeyev (2010; 2013; 2017; 2021) draws an analogy between our usage of infinity and a numeral system used by an Amazonian tribe called "Pirahã" (see Gordon, 2004). Pirahã's numeral system consists only of numbers 1 and 2. Any quantity greater than two is expressed as "many". Due to this weak numeral system, they cannot distinguish, compare, or make any computations for quantities greater than two. Suppose we exchange the "many" by "∞", we will arrive at cases we usually encounter with infinity.

For instance:

"Many" + 1= "many" ; "many" + "many" = "many" ;

"many" – "many" is an indeterminate form

By exchanging "many" by "∞", we get:

[4] The difference between a numeral and a number is that a numeral is a symbol whereas a number is a concept. Numerals are symbol(s) representing numbers. Different numerals can represent the same number.

$\infty + 1 = \infty$; $\infty + \infty = \infty$; $\infty - \infty$ is an indeterminate form.

In the Arithmetic of Infinity, a new infinite unit called *grossone* is introduced and denoted by ①. ① is the number of elements of set $\mathbb{N}$, which means that for any $n \in \mathbb{N}$, $n \leq$①. An extended set of natural numbers is as follows

$\widehat{\mathbb{N}}$: {1,2, 3, … ①-1, ①, ①+1, ①+2, 2①, 2①+1, …., ①^2-1, ①^2, ①^2+1, ...}

The infinitesimals would obviously be the inverses of the grossone based infinite numbers.

In this extended collection $\widehat{\mathbb{N}}$, we may postulate that the objects in this set obey the formal properties that hold in $\mathbb{N}$ (order, addition, multiplication, exponentiation) (Rizza, 2019).

Unlike the traditional theories that work with infinities and infinitesimals as symbols, this methodology allows infinite and infinitesimal numbers to be written in a positional numeral system with infinite radix. This numeral system having base grossone is an extension of the system in base ten. Hence, the numbers in base ① can be written in an analogous manner to that of base 10. For instance, ① +3 corresponds to the base ① record: 1①^1 3①^0 (Rizza, 2019).

This methodology launches from the postulate that the part-whole relation is preserved for infinite collections as well. The grossone numeral system allows easy and intuitive computations of infinite quantities (it is so similar to computation done at the level of finite numbers) and provides an accurate distinction of infinite quantities, which is not the case in classical set theory approach. As a consequence, using this numeral system will eliminate the indeterminate forms usually encountered in the classical approach and avoid many paradoxical cases related to infinities and infinitesimals [5](Sergeyev, 2010; 2013; 2017; 2021; Caldarola, Cortese, d'Atri, & Maiolo, 2020).

Sergeyev (2017) illustrates his idea by using the example of counting number of grains in a granary. Since no one has the ability to count the grains one by one, people use sacks and fill them with seeds and count the number of sacks (it is supposed that all sack can contain the same unknown number of seeds). If the number of sacks is too large and there is no ability to count them, motor lorries or train wagons are used (again, it is supposed that all motor lorries can contain the same unknown number of sacks, etc.). For instance, the granary contains 11 wagons, 10 motor lorries, 8 sacks and 49 seeds.

It is worth mentioning that a computing software for the grossone based system which works numerically with ① has been developed. It is called "*the Infinity Computer*" (Sergeyev, 2017; 2021).

Theoretical Background

In this section, the theoretical background used in the study is described.

Beliefs and Changes in Beliefs

D'Amore & Fandiño Pinilla (2004) defined belief as "an opinion, set of judgements/expectations, what is thought about something.

Someone's set of beliefs (A) about something (T) gives the conception (K) of A relative to T. If A belongs to a social group (S) and shares the set of beliefs relative to T with the other members of S, then K is the conception of S relative to T."

[5] For detailed information visit: http://theinfinitycomputer.com

Projecting on our study, (T) is the concept of infinity, (S) is the group of students considered, (A) is each student's belief(s) about infinity which give the conception (K) of infinity.

According to Schoenfeld (1992), knowledge of mathematics and beliefs on mathematics cannot be separated (in Sbaragli, 2006). Beliefs are hidden factors that could affect students' responses to mathematical situations and therefore would affect their achievement as well as their motivation (Eleftherios & Theodosios, 2007). Muis (2004) & Depaepe et al. (2016) found that beliefs regarding mathematics abstract learning instead of facilitating it, which follows that studying these beliefs and trying to reform them is of great importance (in Iannone et al., 2018).

Sample and Setting

A group of 25 grade 12 students (aged 17-19) from the general sciences and life sciences section[6] were arbitrary chosen from two official schools in Lebanon (Ain-Zahalta and Majdelbaana secondary official schools). Official schools were chosen for this study in order to ensure that the Lebanese curriculum is adopted. A permission from the Lebanese ministry of education was requested in order to implement the study.

The study was implemented at the end of the academic year 2020/2021. It is noted that the author was the mathematics teacher of grade 12 in Ain-Zahalta school during 2020/2021.

The choice of grade 12 students goes back to that they have encountered the infinity concept and dealt with it in the classical approach. This allows them to compare the two methodologies and give a feedback on the new one.

[6] General science and life science section in the Lebanese curriculum take mathematics as a basic subject.

It is noted that usually grade 12 students studying the Lebanese curriculum are subject to basic calculus (study of functions, sequences, integrals, …), basic probability, geometry (conics, transformations, analytic geometry…), complex numbers, basic logic[7]. Nevertheless, due to circumstances related to the Covid-19, some topics (conics, logic, sequences, higher order derivatives, metric relations…) were suspended for the academic year 2020/2021[8].

Method

The study was done in three phases. In phase 1, students filled a questionnaire (pre-questionnaire) regarding their beliefs about mathematical infinity and the obstacles related to it. In phase 2, students were exposed to a session introducing grossone methodology after which they performed an evaluation and a post-questionnaire in phase 3.

The aim of the pre/post- questionnaires was to check students' beliefs and attitudes towards the mathematical infinity and sizes of infinite sets before and after the intervention.

The aim of the intervention was to introduce the new methodology and help students work out computations and basic arithmetic relying on grossone notion[9]. In particular, this intervention aims at checking the efficiency of this methodology in comparing infinite sets.

The test at the end of the intervention was held to check whether students are able to deal successfully with infinity using the new approach.

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[7] For more details, check: http://www.crdp.org

[8] To check the amendments, visit: http://www.crdp.org

[9] For content of the session visit the site: http://www.numericalinfinities.com.

Results and Discussion

Phase 1:

In the pre-questionnaire (see Appendix), students were given statements in which they have to tell if they are convinced or not about them. Students' beliefs were requested in this questionnaire not to fall in the trap of what they know or what they have been taught.

The results are summarized in the table below:

Table 1: Results of question 1 of the pre-questionnaire

Statement	Percentage of students who are convinced about it
$0.\overline{9}=1$	56%
$\mathbb{N}$ and $\mathbb{Z}$ have the same size	20%
There are many sizes of infinity	56%
There are even numbers as many as there are odd numbers	80%
$\{1,3,5,\ldots\}$&$\mathbb{N}$ have the same number of elements	28%
There are as many rational numbers as there are natural numbers	24%
There are more real numbers than rational numbers	72%

As clearly shown and as expected, the statements that are in line with students' logic and intuitions are highly accepted. For example, it is natural to think that the number of evens and odds is the same and that there are more real numbers than there are rational numbers. On the other hand, results show that 20% believe that there are natural numbers as many as there are integers and also 28% only believe that the set of odd numbers and the set of natural numbers have the same size.

The struggle in comparison of infinite sets seemed obvious from the students' responses. Some students crossed then changed their answers and one student wrote "I don't know" next to several statements. There were also responses that hold contradictions between a statement and another, which shows the state of unclarity and ambiguity faced in comparison of infinite sets.

In the same questionnaire, students were asked to write their attitudes and beliefs when they deal with a problem situation involving infinity. Most of the students used negative words that reflect their discomfort and vague thoughts of this concept.

Here are some phrases that appeared many times: *hard, complicated, not clear, vague, weird, abstract, tough, puzzling, beyond man's understanding, sometimes like numbers others not, always afraid with infinity…*

Few students seem to have a more confident attitude, but still they find it abstract and hard to explain. One student wrote: "*It is hard, but we have techniques to deal with it like sum of infinite geometric sequence and Hopital's rule for some indeterminate limits.*"

Other responses: *"When I use infinity as number it becomes easy.";*

"*Sometimes I get correct answer but I am not convinced.";*

"Though we have ways to deal with infinity, it is still not clear."

We can tell from the above responses that even if students find their way out in situations involving infinity, they are still not convinced and have unclear beliefs and negative attitudes. According to our theoretical background, beliefs lead to the formation of conceptions. And in this case, their conceptions of infinity will lack coherence and then it would be hard to reform or change them.

These results were expected and align with many other studies (Aztekin, Arikan & Sriraman, 2010; Schwarzenberger & Tall, 1978; Tall, 1980; Weller et al., 2009; Kattou et al., 2010; Nasr, 2015). It is because the way we deal with this concept is in opposition with the natural and intuitive human thinking.

In the last part of the questionnaire, students were asked whether they wish there were an arithmetic to compute infinity similar to that used for finite numbers. 88% of the sample wrote "yes". This reveals that students are in need to such an arithmetic and are open to new approaches. One of the students who answered "no" was asked about his response during the class discussion, and he responded that he doesn't believe the existence of such an arithmetic is possible because infinity is so abstract and beyond man's reach.

Phase 2:

In this phase, the students (divided to three groups) were exposed to the grossone methodology, in a two-hour session. The methodology was explained in brief and students worked out with some computations and arithmetic.

Most students seemed to be very welcoming to the new methodology. They found it convincing and logical. They enjoyed the computations and were surprised by its simplicity and ease. It was noted that the students were able to grasp, understand and workout with grossone in a relatively short period of time. This result is compatible with other studies done on this methodology (Antoniotti, Caldarola, d'Atri & Pellegrini, 2020; Iannone et al., 2018, Ingarozza et al., 2020).

An interesting discussion was held. Voices of rejection and doubts and others of acceptance.

Haitham: "*You haven't solved the problem since we don't know what grossone equals like we don't know what infinity is.*"

Petra: *"Even if we don't have a value for grossone, but still it works as a reference to compare different infinities with, like the example of the sacks containing grains."*

Tala: "*How can we talk about last element in the set of natural numbers if this set has no last element?"*

Dany: *"You are changing in mathematics. Can we do this?"*

Dany evoked an idea that is of great importance. Many students believe that mathematics is static and cannot be altered or that changes are not allowed. This line of thinking also shows in a study done on secondary students where Iannone et al. (2018) distinguished between different epistemologies of mathematics that students have. Those who believe that mathematics is fixed and cannot be changed and others who accept changes and consider mathematics as a result of social construct.

Phase 3:

In this phase, the students performed a 30-minute evaluation. The evaluation involved computations with grossone and comparisons of infinite sets using grossone methodology[10]. The results were as follows:

[10]The questions of the evaluation (see Appendix) were simulated from the manual prepared by Davide Rizza. Check: http://www.numericalinfinities.com

Table 2: Results of the evaluation

% of correct answers	Less than 60%	Between 60% &80%	More than 80%
% of students	8%	60%	32%

The results of the evaluation show that 92% of the sample were able to pass the evaluation (> 60% correct answers). It is noted that high achievers in mathematics performed better than low achievers. This result could be because the approach is similar to computations done on finite numbers. Another reason could be that high achievers have good logical thinking and have ability to grasp new ideas in a short period of time. Nevertheless, even students who are average in their mathematics classes were able to solve correctly more than 60% of the evaluation.

These results come after just a two–hour session. Maybe if more time had been given and more worksheets had been solved in class, the results would have been even better.

In this phase, students were also asked to fill a questionnaire which requests their reflections and their attitudes towards the mathematical infinity after being exposed to the Arithmetic of Infinity.

Here are some phrases that appeared multiple times: "*easy, clear, logical and more accurate.*"

"grossone simplified work with infinity."

"no more indeterminate forms with this new method."

"now comparison of infinite sets is so easy."

"very new, we need time to get used to it but it is easy"

"we need more practice because this is different from what we have learned."

Comparing with the pre-questionnaire, the shift of students' attitudes towards infinity is well noticed. All the negative words that students had used in the pre-questionnaire to describe their beliefs and attitudes of the mathematical infinity are now replaced by positive words that show comfort and more confidence in dealing with infinity. Probably, the students had this welcoming attitude to the new methodology, because they have dealt with the traditional methods with all the complications and paradoxes related to infinity.

Concluding, students' beliefs of the mathematical infinity are a result of their previous experiences with it. These beliefs could have a negative effect on their performance and learning. So in order to change their beliefs, providing an alternative approach could be the factor that would make this change happen. This paper tested the effect of *the Arithmetic of Infinity* methodology on students' beliefs and has proved to be promising. Further investigations of the effect of this methodology are needed on a wider sample and various topics related to infinity.

Conflict of Interest Statement

The author states that there is no conflict of interest and no funding has been used to execute this research.

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Appendix 1:

Pre-Questionnaire

This survey is designed to gather information for sake of educational research. Your answers will remain confidential and will be used for the purpose of the research only. The names will not appear in the research as well. The survey takes no more than 10 minutes to complete. Please read and answer the questions well for accuracy of the results. Thank you for your time!

Name: ………………

Section: ……………..

I. Answer "true" if the phrases below convince you and "false" otherwise:

1) $0.999... = 1$
2) The set of natural numbers $\mathbb{N}$ and the set of integers $\mathbb{Z}$ have the same size.
3) There are many sizes of infinity.
4) There are even numbers as much as there are odd numbers.
5) $\{1,3,5,7, ...\}$ & $\mathbb{N}$ have the same number of elements.
6) There are as many rational numbers as there are natural numbers.
7) There are more real numbers than rational numbers.

II. Describe in your own words your attitudes and beliefs of mathematical infinity.

…………………………………………………………………………………………………

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………………………………………………

III. Do you wish if there is an arithmetic to compute infinity similar to that we use with finite numbers? Answer by “yes” or “no”

Appendix 2

Evaluation

This test is designed to gather information for sake of educational research. Your answers will remain confidential and will be used for the purpose of the research only. The names will not appear in the research as well. The duration of this test 30 minutes. Please read and answer the questions well for accuracy of the results. Thank you for your time!

Name: …………………….

Section: …………………..

I. (4 pts.)

Write the following terms as ① records.

1) 345
2) 2 ① + 4
3) 3 $①^3$+ 43 ① + 354
4) 3- 1/ $①^4$

II. (6 pts.)

Simplify the following terms:

1) 12(① + 3)- 6 (3 ① + 6)

2) (① + 1)² - ① (4 ① + 2)

3) ① [5 ①² - 2 ① + 3) – 2(①² + 4 ①) - 3]

4) (① /2 - ① /6) [(2/ ①) +2 ① - 4]

III. (6 pts.)

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1) Determine the number of elements of $\mathbb{N}_{2,3}$.

2) $A = \{3,6,9,12,15,18,\dots\}$ & $B = \{2,6,10,14,18,\dots\}$

   Determine the number of elements of the set $A \cap B \cup \{3,6,9,10,11\}$.

3) Compare a completed count of $\mathbb{N}$ and one of $\mathbb{N}_{2,5}$.

IV. (4 pts.)

Determine whether each of the numbers is even or odd and whether it is in $\mathbb{N}$ or not:

1) ① /7

2) ① - ① /4

Appendix 3

Post-Questionnaire

This survey is designed to gather information for sake of educational research. Your answers will remain confidential and will be used for the purpose of the research only. The names will not appear in the research as well. The survey takes no more than 5 minutes to complete. Please read and answer the question well for accuracy of the results. Thank you for your time!

Name: ………………

Section: ……………..

After being exposed to the grossone methodology, what are your reflections of it. Describe your attitudes towards infinity now.

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